A) Generalized coordinates and constraints

In example; possible motion of m and M is restricted, e.g., M can only move along floor, m is "held up" by M.
These restrictions are called "<u>constraints</u>"
example: motion on surface of sphere.

How to describe constraints:

(1) "holonomic constraints":
$f_k(\underline{r}_1, \underline{r}_2 \ldots r_n; t) = 0$, i.e. k ~~eqns~~ constraints, expressed as equalities.

e.g. pendulum $f = x^2 + y^2 - l^2 = 0$

rigid body $f_{ij} = (\underline{r}_i - \underline{r}_j)^2 - c_{ij}^2 = 0$

otherwise "nonholonomic"

e.g. gas inside sphere: $\underline{r}^2 - a^2 < 0$
(expressed as <u>inequality</u>)

12) time dependence of constraints:

explicitly time dep: "rheonomic c's"

not — " — "scleronomic c's"

Most important class:

<u>scleronomic</u>, <u>holonomic</u> constraints

usually it is possible to express constraint as

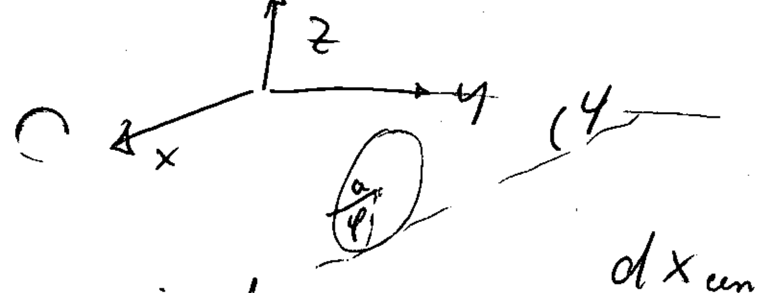
$$(*) \quad \sum_k a_k (x_1, x_2 \ldots) \, dx_k = 0$$

holonomic iff $\dfrac{\partial a_k}{\partial x_i} = \dfrac{\partial a_i}{\partial x_k}$

(proof: $a_k \equiv \dfrac{\partial U}{\partial x_k} \Rightarrow \sum_k a_k \, dx_k = dU$

$\Rightarrow$ integrable into $U = 0$

- counterexample: wheel on plane:



$v_{center} = a \dot{\varphi}$

$\dot{x}_{center} = - v_{center} \sin \psi$

$\dot{y}_{center} = v_{center} \cos \psi$

in form $(*)$: $dx_{center} + a \sin \psi \, d\varphi = 0$

$dy_{center} - a \cos \psi \, d\varphi = 0$

not integrable.

<u>Problem:</u> coordinates with constraints are <u>not</u> independent.

$\Rightarrow$ system of N masses has

$$\begin{array}{rl} & 3N \text{ coordinates} \\ \text{and} & -\ s \text{ constraints} \\ \hline & 3N - s \text{ degrees of freedom.} \end{array}$$

ex: solid bodies without constraints: 5 coordinates
(x, y, z for COM
ψ, φ around COM)

in  :

 motion is constrained to x only $\Rightarrow$ no y, z, φ, ψ

φ  (no slipping)
φ only $\Rightarrow$ no (other) x, ~~z~~, no φ, no ψ

$\Rightarrow$ only <u>two</u> DOF.

$\Rightarrow \quad \underline{r}_1 = \underline{r}_1(q_1, \ldots, q_{3N-s}; t)$

$\underline{r}_2 = \ldots$

q_α are "generalized coordinates"
(e.g. distances, angles, ...)

example: cylinder (gliding and rolling) on plane: dof:

l
φ

thus: $\dot{\underline{r}}_i = \sum_\alpha \frac{\partial \underline{r}_i}{\partial q_\alpha} \dot{q}_\alpha + \frac{\partial \underline{r}_i}{\partial t}$

$\left(\text{and } \frac{\partial \dot{\underline{r}}_i}{\partial \dot{q}_\beta} = \frac{\partial \underline{r}_i}{\partial q_\beta} \right)$

$$\Rightarrow dW = \sum_{i=1}^{3N} \underline{F}_i \cdot d\underline{r}_i = \sum_{\alpha=1}^{3N-s} \underbrace{\left(\sum_i \underline{F}_i \frac{\partial \underline{r}_i}{\partial q_\alpha} \right)}_{\equiv Q_\alpha} dq_\alpha$$

"generalized force"

conservative system: (no expl. time dep. in W)

$$dW = \sum_\alpha \frac{\partial W}{\partial q_\alpha} dq_\alpha \Rightarrow \boxed{Q_\alpha = \frac{\partial W}{\partial q_\alpha}}$$

3) D'Alembert's principle

"virtual displacement" $\delta \underline{r}$: infinitesimal displacement in accordance with constraints

(but! forces and constraints do not change with $\delta \underline{r}$!)

[calculate like with $d\underline{r}$]

Basic idea:

Constraint forces do not perform work!

$$\Rightarrow \sum_i \underbrace{\underline{F}_i^{z}}_{\text{constraint forces}} \cdot \delta \underline{r}_i = 0$$

with $\underline{F}_i = \underline{F}_i^{z} + \underline{F}_i^{(e)}$,

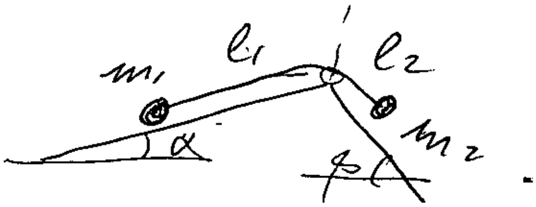
and $\sum_i (\underline{F}_i - \dot{\underline{p}}_i)\, \delta \underline{r}_i = 0$

$$\boxed{\sum_i (\underline{F}_i^{e} - \dot{\underline{p}}_i)\, \delta \underline{r}_i = 0}$$ d'Alembert's principle

(usually $\underline{F}_i^{z} \perp \delta \underline{r}_i$

$\Rightarrow \quad (\underline{F}_i^{e} - \dot{\underline{p}}_i) \cdot \delta \underline{r}_i = 0$)

- example:



two masses on string –

d'Al.: $(\underline{F}_1 - \dot{\underline{p}}_1)\,\delta \underline{l}_1 + (\underline{F}_2 - \dot{\underline{p}}_2)\,\delta \underline{l}_2 = 0$

constraint: $|\underline{l}_1| + |\underline{l}_2| = l$

$$\left[\Rightarrow \ddot{l}_1 = \frac{m_1 \sin\alpha - m_2 \sin\beta}{m_1 + m_2}\, g \right]$$

- problem: needs both d'Al. eqs + constraints

- in generalized coordinates → Lagrange's eqs

$$\sum_i \underline{F}_i \cdot \delta \underline{r}_i = \sum_\alpha Q_\alpha \, \delta q_\alpha \qquad (1)$$

and $\sum_i \dot{\underline{p}}_i \, \delta \underline{r}_i =$

$$= \sum_{i,\beta} \Big[\frac{d}{dt} (m_i \dot{\underline{r}}_i) \cdot \frac{\partial \underline{r}_i}{\partial q_\beta} \delta q_\beta + \Big(m_i \dot{\underline{r}}_i \frac{d}{dt} \frac{\partial \underline{r}_i}{\partial q_\beta} - m_i \dot{\underline{r}}_i \frac{d}{dt} \frac{\partial \underline{r}_i}{\partial q_\beta} \Big) \delta q_\beta \Big]$$