B) Systems of particles (how does a cow fly?)

Distinguish two types of forces:

- external: $\underline{F}_i^{(e)}$ (external force on particle i)
- internal: $\underline{F}_{ji}$ (internal force from particle j on part. i)

$$\underline{F}_i^{\text{total}} = \sum_{j(\neq i)} \underline{F}_{ji} + \underline{F}_i^{(e)} = \dot{\underline{p}}_i = m_i \ddot{\underline{r}}_i$$

whole system: $\frac{d^2}{dt^2} \sum_i m_i \underline{r}_i = \sum_i \underline{F}_i^{(e)} + \sum_{\substack{j,i \\ (j \neq i)}} \underline{F}_{ji}$

$$\left(\text{with } \underline{F}_{ij} = -\underline{F}_{ji} \Rightarrow \sum_{\substack{j,i \\ (j \neq i)}} \underline{F}_{ji} = 0 \right)$$

$$\Rightarrow M \ddot{\underline{R}} = \sum_i \underline{F}_i^{(e)} = \underline{F}^{(e)}$$

where $\underline{R} = \frac{\sum_i m_i \underline{r}_i}{\sum m_i} = \frac{1}{M} \sum_i m_i \underline{r}_i$

momentum: $\underline{P} = M \dot{\underline{R}} = \sum_i m_i \dot{\underline{r}}_i$

$\Rightarrow$ 1st conservation theorem:

> If total external force is zero, the total linear momentum $\underline{P}$ is conserved.

angular momentum: $\underline{L} = \sum_i \underline{r}_i \times \underline{p}_i$

$$\dot{\underline{L}} = \frac{d}{dt} \sum_i \underline{r}_i \times \underline{p}_i = \sum_i \cancelto{0}{\underline{v}_i \times m_i \underline{v}_i} + \sum_i \underline{r}_i \times \dot{\underline{p}}_i$$

$$= \sum_i (\underline{r}_i \times \underline{F}_i^{(e)}) + \sum_{\substack{j,i \\ (j \neq i)}} \cancelto{0}{(\underline{r}_i \times \underline{F}_{ji})}$$

(because $\underline{r}_i \times \underline{F}_{ji} + \underline{r}_j \times \underline{F}_{ij} = (\underline{r}_i - \underline{r}_j) \times \underline{F}_{ji}$

$= 0 \quad (\underline{r}_i - \underline{r}_j \parallel \underline{F}_{ij})$

$\Rightarrow \dot{\underline{L}} = \underline{N}^{(e)} \quad (= \sum_i \underline{N}_i^{(e)})$

$\Rightarrow$ 2nd conservation theorem

If total external torque is zero, the total angular momentum $\underline{L}$ is conserved.

Center-of-mass motion (COM)

Def: $\underline{R}$: center-of-mass coord.

$$\Rightarrow \underline{r}_i = \underline{R} + \underline{r}_i'$$
$$\underline{v}_i = \dot{\underline{R}} + \underline{v}_i' \qquad (\underline{v}_i' = \dot{\underline{r}}_i')$$

angular mom: $\underline{L} \overset{= \sum l_i}{=} \sum_i (\underline{R} + \underline{r}_i') \times m_i (\dot{\underline{R}} + \underline{v}_i')$

$$= M\underline{R} \times \dot{\underline{R}} + \sum_i \underline{r}_i' \times m_i \underline{v}_i' + \cancel{\left(\sum m_i \underline{r}_i'\right)}^{0} \times \dot{\underline{R}} + \underline{R} \times \frac{d}{dt} \cancel{\sum_i (m_i \underline{r}_i')}^{0}$$

$$= \underline{R} \times \underline{P} + \sum_i \underline{r}_i' \times \underline{p}_i'$$

Angular momentum =

angular mom of COM motion + ang. mom. around COM

<u>energy</u>: $W_{12} = \int_1^2 \underline{F}_i \cdot \underline{ds}_i$

(1, 2 are configurations)

$= \sum_i \int_1^2 \frac{d}{dt}\left(\frac{1}{2} m_i v_i^2\right) dt = T_2 - T_1$ is valid

with $T = \frac{1}{2} \sum_i m_i v_i^2 =$

$= \frac{1}{2} \sum_i m_i \left(\dot{\underline{R}} + \underline{v}_i'\right) \cdot \left(\dot{\underline{R}} + \underline{v}_i'\right) =$

$= \frac{1}{2} M \dot{\underline{R}}^2 + \frac{1}{2} \sum_i m_i v_i'^2 + \frac{1}{2} \cdot 2 \cdot \dot{\underline{R}} \frac{d}{dt}\left(\sum_i m_i \underline{r}_i'\right)^{0}$

$= T_{com} + T_{rel}$

<u>Conservative forces</u>:

- $\underline{F}_i^{(e)}$: $\sum_i \int_1^2 \underline{F}_i^{(e)} \cdot \underline{ds} = - \sum_i V_i \Big|_1^2$
- $\underline{F}_{ji} = -\nabla_i V_{ij} = -\nabla_{ij} V_{ij} = +\nabla_j V_{ij} =$

$V_{ij} = V(|\underline{r}_i - r_j|) = V_{ji}$

$\sum\limits_{\substack{j,i\\(j\neq i)}} \int_1^2 \underline{F}_{ji} \cdot \underline{ds}_i$ contains

$$-\int_1^2 \left(\underline{\nabla}_i V_{ij} \cdot \underline{ds}_i + \underline{\nabla}_j V_{ji} \cdot \underline{ds}_j \right) =$$

$$= -\int_1^2 \underline{\nabla}_{ij} V_{ij} \cdot (d\,\underline{s}_j - d\underline{s}_i) = -\int_1^2 \underline{\nabla}_{ij} V_{ij} \cdot d\underline{s}_{ij}$$

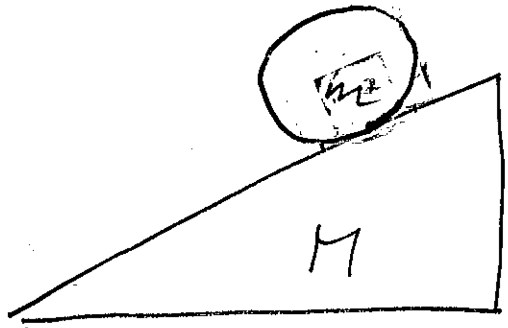
$$\Rightarrow \sum_{\substack{i,j\\(j\neq i)}} \int_1^2 \underline{F}_{ji} \cdot \underline{ds}_i = -\frac{1}{2} \sum_{\substack{i,j\\(j\neq i)}} V_{ij} \Big|_1^2$$

<u>potential</u>: $V = \sum\limits_i V_i + \frac{1}{2} \sum\limits_{\substack{j,i\\(j\neq i)}} V_{ji}$

"external potential energy" "internal potential energy"

II. Lagrange's Equations

Example:



One cylinder and one block (masses m and M) can slide* frictionless. M and m start at rest. Let them go – how fast will M move once m hits the floor?

How to solve?

- which variables? how many?
- which forces between m and M, M and floor?
- how to formulate equations of motion?

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* or roll