P1

Notations $u(j\Delta x, n\Delta t) = u_j^n$

The finite difference scheme:

$$(*) \quad u_j^{n+1} - u_j^n + \mu\left(-\frac{1}{2}u_{j-2}^n + u_{j-1}^n - u_{j+1}^n + \frac{1}{2}u_{j+2}^n\right) = 0$$

where $\mu = \dfrac{\Delta t}{\Delta x^3}$, $\mu > 0$

Von Neuman stability analysis: $i = \sqrt{-1}$

$$u_{j\pm1}^n = e^{\pm ik\Delta x} u_j^n \;; \qquad u_{j\pm2}^n = e^{\pm 2ik\Delta x} u_j^n$$

Plug those relations into $(*)$:

$$u_j^{n+1} = u_j^n - \mu\Big(\frac{1}{2}\underbrace{\left(e^{2ik\Delta x} - e^{-2ik\Delta x}\right)}_{2i\sin(2k\Delta x)} - \underbrace{\left(e^{ik\Delta x} - e^{-ik\Delta x}\right)}_{2i\sin(k\Delta x)}\Big)u_j^n$$

$$u_j^{n+1} = u_j^n \underbrace{\Big[1 - i\mu\left(\sin(2k\Delta x) - 2\sin(k\Delta x)\right)\Big]}_{G}$$

$$\boxed{u_j^{n+1} = G\, u_j^n}$$

← The finite difference scheme is unconditionally stable if $|G| \leq 1$ for any μ, ~~unstab~~ unconditionally unstable if $|G| > 1$ for any value of μ.

In 'our' case

$$G = 1 + i\mu\, \delta(k)$$

where $\delta(k) = 2\sin(k\Delta x) - \sin(2k\Delta x) \not\equiv 0$

$$|G|^2 = 1 + \mu^2 \delta^2(k) > 1$$

thus $|G| > 0$, ~~and then~~ for any value of μ, $\longrightarrow$ the scheme is unconditionally unstable.

P2

(a) (*) $\theta'' + p\sin\theta = 0$, $\theta = \theta(s)$

where prime denotes derivative with respect to s.

$z = \dfrac{d\theta}{dt}$, where $t = \theta'(0)$;

~~deri~~ We can change the order of derivatives with respect to t and to s.

Take the derivative with respect to t from both sides of (*)

$$\frac{d}{dt}\theta'' = \frac{d}{dt}\frac{d^2}{ds^2}\theta = \frac{d^2}{ds^2}\underbrace{\frac{d}{dt}\theta}_{z} = \frac{d^2}{ds^2}z = z''$$

$$\frac{d}{dt}\sin\theta = \cos\theta \cdot \underbrace{\frac{d\theta}{dt}}_{z} = z\cdot\cos\theta$$

Thus, the equation

(**) $z'' + p\cdot z\cdot\cos\theta = 0$

(*) and (**) have to be solved together

P 3

(a) $u_e(x, y) = \sin(x \cdot y)$

$$\frac{\partial u_e}{\partial x} = y\cos(xy); \quad \frac{\partial^2 u_e}{\partial x^2} = -y^2 \sin(xy)$$

Similarly

$$\frac{\partial^2 u_e}{\partial y^2} = -x^2 \sin(xy)$$

Thus

$$\frac{\partial^2 u_e}{\partial x^2} + \frac{\partial^2 u_e}{\partial y^2} = -(x^2+y^2) \cdot \sin(xy)$$

which is our equation