P1 Rayleigh quotient

(a) $$\lambda = \frac{\vec{v}^t A \vec{v}}{|\vec{v}|^2}$$

$$|\vec{v}|^2 = 2^2 + 3^2 + 1^2 = 14$$

$$\vec{v}^t A \vec{v} = \begin{pmatrix} 2 & -3 & -1 \end{pmatrix} \begin{pmatrix} -2 & -4 & 2 \\ -2 & 1 & 2 \\ 4 & 2 & 5 \end{pmatrix} \begin{pmatrix} 2 \\ -3 \\ -1 \end{pmatrix}$$

$$= \begin{pmatrix} 2 & -3 & -1 \end{pmatrix} \begin{pmatrix} -4+12-2 \\ -4-3-2 \\ 8-6-5 \end{pmatrix} = \begin{pmatrix} 2 & -3 & -1 \end{pmatrix} \begin{pmatrix} 6 \\ -9 \\ -3 \end{pmatrix}$$

$$= 12 + 27 + 3 = 42$$

$$\boxed{\lambda = \frac{42}{14} = 3}$$

(b) $$B = A - \frac{\lambda}{|\vec{v}|^2} \vec{v} \cdot \vec{v}^t$$

$$\frac{\lambda}{|\vec{v}|^2} \vec{v} \cdot \vec{v}^t = \frac{3}{14} \begin{pmatrix} 2 \\ -3 \\ -1 \end{pmatrix} \begin{pmatrix} 2 & -3 & -1 \end{pmatrix} =$$

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$$= \frac{3}{14}\begin{pmatrix} 4 & -6 & -2 \\ -6 & 9 & 3 \\ -2 & 3 & 1 \end{pmatrix} = \begin{pmatrix} \frac{12}{14} & -\frac{18}{14} & -\frac{6}{14} \\ -\frac{18}{14} & \frac{27}{14} & \frac{9}{14} \\ -\frac{6}{14} & \frac{9}{14} & \frac{3}{14} \end{pmatrix}$$

$$B = \begin{pmatrix} -\frac{28}{14} & -\frac{56}{14} & \frac{28}{14} \\ -\frac{28}{14} & \frac{14}{14} & \frac{28}{14} \\ \frac{56}{14} & \frac{28}{14} & \frac{70}{14} \end{pmatrix} - \begin{pmatrix} \frac{12}{14} & -\frac{18}{14} & -\frac{6}{14} \\ -\frac{18}{14} & \frac{27}{14} & \frac{9}{14} \\ -\frac{6}{14} & \frac{9}{14} & \frac{3}{14} \end{pmatrix}$$

$$= \frac{1}{14}\begin{pmatrix} -40 & -38 & 34 \\ -10 & -13 & 19 \\ 62 & 19 & 67 \end{pmatrix}$$

P2

$$A = \begin{array}{ccc} \begin{pmatrix} 2 & -2 & 18 \\ 2 & 1 & 0 \\ 1 & 2 & 0 \end{pmatrix} \\ \vec{a}_1 \quad \vec{a}_2 \quad \vec{a}_3 \end{array}$$

(*) $\vec{u}_1 = \vec{a}_1$; $\vec{v}_1 = \frac{\vec{u}_1}{|\vec{u}_1|} = \frac{1}{\sqrt{2^2+2^2+1}} = \frac{1}{3}\begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 2/3 \\ 2/3 \\ 1/3 \end{pmatrix}$

$r_{11} = |\vec{u}_1| = 3$

(**) $\vec{u}_2 = \vec{a}_2 - (\vec{v}_1 \cdot \vec{a}_2) \cdot \vec{v}_1$;

$\vec{v}_1 \cdot \vec{a}_2 = r_{12} = \begin{pmatrix} 2/3 & 2/3 & 1/3 \end{pmatrix} \begin{pmatrix} -2 \\ 1 \\ 2 \end{pmatrix} = -\frac{4}{3} + \frac{2}{3} + \frac{2}{3} = 0$

$\vec{v}_2 = \frac{1}{|\vec{u}_2|} \vec{u}_2 = \frac{1}{\sqrt{2^2+1+2^2}} \begin{pmatrix} -2 \\ 1 \\ 2 \end{pmatrix} = \begin{pmatrix} -2/3 \\ 1/3 \\ 2/3 \end{pmatrix}$

$r_{22} = |\vec{u}_2| = 3$

(***)

$\vec{u}_3 = \vec{a}_3 - (\vec{v}_1 \cdot \vec{a}_3) \cdot \vec{v}_1 - (\vec{v}_2 \cdot \vec{a}_3) \cdot \vec{v}_2$

$r_{13} = \vec{v}_1 \cdot \vec{a}_3 = \begin{pmatrix} 2/3 & 2/3 & 1/3 \end{pmatrix} \begin{pmatrix} 18 \\ 0 \\ 0 \end{pmatrix} = 12$

$r_{23} = \vec{v}_2 \cdot \vec{a}_3 = \begin{pmatrix} -2/3 & 1/3 & 2/3 \end{pmatrix} \begin{pmatrix} 18 \\ 0 \\ 0 \end{pmatrix} = -12$

$$\vec{u}_3 = \begin{pmatrix} 18 \\ 0 \\ 0 \end{pmatrix} - 12 \begin{pmatrix} 2/3 \\ 2/3 \\ 1/3 \end{pmatrix} + 12 \begin{pmatrix} -2/3 \\ 1/3 \\ 2/3 \end{pmatrix}$$

$$= \begin{pmatrix} 18 \\ 0 \\ 0 \end{pmatrix} - \begin{pmatrix} 8 \\ 8 \\ 4 \end{pmatrix} + \begin{pmatrix} -8 \\ 4 \\ 8 \end{pmatrix} = \begin{pmatrix} 2 \\ -4 \\ 4 \end{pmatrix}$$

$$r_{33} = |\vec{u}_3| = \sqrt{2^2 + 4^2 + 4^2} = 6$$

$$\vec{v}_3 = \frac{1}{|\vec{u}_3|} \vec{u}_3 = \begin{pmatrix} 1/3 \\ -2/3 \\ 2/3 \end{pmatrix}$$

$$A = QR$$

$$Q = (\vec{v}_1 \ \vec{v}_2 \ \vec{v}_3) = \begin{pmatrix} 2/3 & -2/3 & 1/3 \\ 2/3 & 1/3 & -2/3 \\ 1/3 & 2/3 & 2/3 \end{pmatrix}$$

$$R = \begin{pmatrix} 3 & 0 & 12 \\ 0 & 3 & -12 \\ 0 & 0 & 6 \end{pmatrix}$$

$QR =$

$$= \begin{pmatrix} 2/3 & -2/3 & 1/3 \\ 2/3 & 1/3 & -2/3 \\ 1/3 & 2/3 & 2/3 \end{pmatrix} \begin{pmatrix} 3 & 0 & 12 \\ 0 & 3 & -12 \\ 0 & 0 & 6 \end{pmatrix} =$$

$$= \begin{pmatrix} 2 & -2 & 8+8+2 \\ 2 & 1 & 8-4-4 \\ 1 & 2 & 4-8+4 \end{pmatrix} = \begin{pmatrix} 2 & -2 & 18 \\ 2 & 1 & 0 \\ 1 & 2 & 0 \end{pmatrix} = A$$

$$\vec{V}_1 \cdot \vec{V}_2 = \begin{pmatrix} 2/3 & 2/3 & 1/3 \end{pmatrix} \begin{pmatrix} -2/3 \\ 1/3 \\ 2/3 \end{pmatrix} = -\frac{4}{9} + \frac{2}{9} + \frac{2}{9} = 0$$

$$\vec{V}_1 \cdot \vec{V}_3 = \begin{pmatrix} 2/3 & 2/3 & 1/3 \end{pmatrix} \begin{pmatrix} 1/3 \\ -2/3 \\ 2/3 \end{pmatrix} = \frac{2}{9} - \frac{4}{9} + \frac{2}{9} = 0$$

$$\vec{V}_2 \cdot \vec{V}_3 = \begin{pmatrix} -2/3 & 1/3 & 2/3 \end{pmatrix} \begin{pmatrix} 1/3 \\ -2/3 \\ 2/3 \end{pmatrix} = -\frac{2}{9} - \frac{2}{9} + \frac{4}{9} = 0$$