Question:	1	2	3	4	5	6	7	Total
Points:	20	15	5	5	15	30	10	100

Show all your work and indicate your reasoning in order to receive the credit. Present your answers in *low-entropy* form.

Fourier transform

1. Find the DFTs of the vector $\mathbf{x} = [0, 1, 1, 1]$:

   (a) (5 points) Write down the Fourier transform matrix F_4. Use the Euler formula $e^{i\phi} = \cos\phi + i\sin\phi$, if needed, to evaluate the exponents of complex numbers.

   (b) (5 points) Find the vector $\mathbf{X} = F_4 \cdot \mathbf{x}$. Show all your work in the space below.

   (c) (5 points) Write down the matrix of the inverse Fourier transform F_4^{-1}.

   (d) (5 points) Find the inverse Fourier transform of the vector $\mathbf{X}$ that you calculated earlier. Verify that you recovered the original vector $\mathbf{x}$. Show all your work.

Vector and matrix norms

2. Find l_2 and l_∞ norms of the vectors.

   (a) (5 points) $x = (12, -5, 0)^t$

   (b) (5 points) $x = (1, 2, 1, 4)^t$

   (c) (5 points) $x = \left(\frac{a}{\sqrt{a^2+b^2}}, \frac{b}{\sqrt{a^2+b^2}}, 2\right)$ for arbitrary real a, b

3. (5 points) Find l_∞ norms of the matrix

$$\begin{bmatrix} 2 & 5 & 0 & 0 & 0 & 0 & 0 & 3 \\ 0 & -5 & 3 & 2 & 4 & 0 & 0 & 0 \\ 0 & 0 & 0 & -2 & 0 & 2 & 1 & 0 \\ 1 & -1 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 1 & -1 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 & -1 & 0 & 1 & 0 \end{bmatrix}$$

4. (5 points) Find l_2 and l_∞ norm of the matrix I^3, where I is identity $n \times n$ matrix.

Eigenvalues and eigenvectors

5. Compute the eigenvalues and associated eigenvectors of the following matrices. Find the spectral radius for each matrix.

(a) (5 points)

$$\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$$

(b) (10 points)

$$\begin{bmatrix} 2 & 1 & 0 \\ 1 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

Matlab programming

6. Compare the convergence rate of the Jacobi method with the theoretical predictions.

   (a) (10 points) write the matlab function that given the matrix A of the system of linear equations $Ax = b$, calculates the theoretical convergence rate:

   ```
   function sigma = jacobitheor(A)
   % JACOBITHEOR theoretical convergence rate for jacobi iterations
   ...
   end
   ```

   (b) (20 points) Verify whether or not the theoretical and experimental convergence rates agree. Conduct your calculations for the following matrices A_i:

   1. `A1 = A;`
   2. `A2 = A + 5*speye(n);`
   3. `A3 = A + 9*speye(n);`

   where `A = laplacian2d(8)` and n is the dimension of the matrix A.

   On the same graph plot the theoretical and experimental convergence parameter vs the iteration number for all calculations. Clearly describe your conclusions in your project's readme file. Place the code you wrote for this part of the exam in the matlab file **m1p6.m**

Git and Gitlab

7. (10 points) Upload the code you wrote for this exam to UConn gitlab server:

   1. Use gitlab web interface to create a new project called **midterm1-sample**. (The name must be exactly as shown.)
   2. Use gitlab web interface to add *README.md* file and edit it to add the relevant content.
   3. Use gitlab web interface to **upload** your matlab code to your project. (Do not copy and paste.)
   4. Use gitlab web interface to grant the access to your project (with the permission of the *reporter*) to the instructor.

P1 $\omega = e^{-\frac{2\pi i}{4}} = e^{-\frac{i\pi}{2}} = -i$

$$F_4 = \begin{pmatrix} \omega^0 & \omega^0 & \omega^0 & \omega^0 \\ \omega^0 & \omega & \omega^2 & \omega^3 \\ \omega^0 & \omega^2 & \omega^4 & \omega^6 \\ \omega^0 & \omega^3 & \omega^6 & \omega^9 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -i & -1 & i \\ 1 & -1 & 1 & -1 \\ 1 & i & -1 & -i \end{pmatrix}$$

$$F_4 \cdot x = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -i & -1 & i \\ 1 & -1 & 1 & -1 \\ 1 & i & -1 & -i \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 3 \\ -1 \\ -1 \\ -1 \end{pmatrix} = X$$

$$F_4^{-1} = \frac{1}{4} (F_4)^* = \frac{1}{4} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & i & -1 & -i \\ 1 & -1 & 1 & -1 \\ 1 & -i & -1 & i \end{pmatrix}$$

$$F_4^{-1} \cdot X = \frac{1}{4} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & i & -1 & -i \\ 1 & -1 & 1 & -1 \\ 1 & -i & -1 & i \end{pmatrix} \begin{pmatrix} 3 \\ -1 \\ -1 \\ -1 \end{pmatrix} = \frac{1}{4} \begin{pmatrix} 0 \\ 4 \\ 4 \\ 4 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 1 \\ 1 \end{pmatrix} = x$$

P2

(a) $x = (12, -5, 0)^t$

$$\|x\|_2 = \sqrt{12^2 + 5^2 + 0} = \sqrt{169} = \boxed{13}$$

$$\|x\|_\infty = \max(12, 5, 0) = \boxed{12}$$

(b) $x = (1, 2, 1, 4)^t$

$$\|x\|_2 = \sqrt{1 + 4 + 1 + 16} = \boxed{\sqrt{22}}$$

$$\|x\|_\infty = \max(1, 2, 4) = 4$$

(c) $x = \left(\frac{a}{\sqrt{a^2+b^2}}, \frac{b}{\sqrt{a^2+b^2}}, 2\right)^t$

$$\|x\|_2 = \sqrt{\underbrace{\frac{a^2}{a^2+b^2} + \frac{b^2}{a^2+b^2}}_{1} + 4} = \boxed{\sqrt{5}}$$

$$\|x\|_\infty = \max\left(\frac{|a|}{\sqrt{a^2+b^2}}, \frac{|b|}{\sqrt{a^2+b^2}}, 2\right) = \boxed{2}$$

P3

$$|A|_\infty = \max(10, 14, \dots) = \boxed{14}$$

P4

$$|I^3|_p = \max_x \frac{|I^3 x|_p}{|x|_p} = \max \frac{|x|_p}{|x|_p} = \boxed{1}$$

P5

(a) $$A = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$$

$$\det(A - I\lambda) = \det \begin{pmatrix} 2-\lambda & -1 \\ -1 & 2-\lambda \end{pmatrix} =$$

$$= (2-\lambda)^2 - 1 = \lambda^2 - 4\lambda + 3 = 0$$

$$\lambda = 2 \pm \sqrt{4-3} = 2 \pm 1$$

$$\lambda_1 = 3; \qquad \lambda_2 = 1$$

Eigenvectors for λ_1:

$$(A - \lambda_1 I)\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0 = \begin{pmatrix} 2-3 & -1 \\ -1 & 2-3 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

$$\begin{pmatrix} -1 & -1 \\ -1 & -1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0 \quad \rightarrow \quad -x_1 - x_2 = 0$$

$$x_1 = -x_2$$

Chose arbitrary x_2, e.g. $x_2 = 1 \rightarrow x_1 = -1$:

$$X = \begin{pmatrix} -1 \\ 1 \end{pmatrix} \quad \text{for} \quad \lambda_1 = 3$$

Eigenvector for λ_2:

$$\begin{pmatrix} 2-1 & -1 \\ -1 & 2-1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0 \quad \rightarrow \quad x_1 - x_2 = 0 \rightarrow x_1 = x_2$$

chose $x_2 = 1 \rightarrow x_1 = 1$

$$X = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad \text{for} \quad \lambda_2 = 1$$

(B) $$A = \begin{pmatrix} 2 & 1 & 0 \\ 1 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}$$

$$\det\begin{pmatrix} 2-\lambda & 1 & 0 \\ 1 & 2-\lambda & 0 \\ 0 & 0 & 3-\lambda \end{pmatrix} = (3-\lambda)\left[(2-\lambda)^2 - 1\right] =$$

$$= -(\lambda-3)(\lambda-3)(\lambda-1)$$

$$\lambda_1 = 1, \quad \lambda_{2,3} = 3$$

Eigenvector for λ_1:

$$\begin{pmatrix} 2-1 & 1 & 0 \\ 1 & 2-1 & 0 \\ 0 & 0 & 3-1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0 \rightarrow \begin{array}{l} x_1 + x_2 = 0 \\ 2x_3 = 0 \end{array}$$

$x_1 = -x_2 \rightarrow x_2 = 1,\ x_1 = -1$

$x_3 = 0$

$$X_1 = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$$

Eigenvectors for $\lambda_{2,3}$:

$$\begin{pmatrix} 2-3 & 1 & 0 \\ 0 & 2-3 & 0 \\ 0 & 0 & 3-3 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0 \qquad \begin{array}{l} -x_1 + x_2 = 0 \\ 0 \cdot x_3 = 0 \end{array}$$

x_3 - arbitrary $\qquad x_1 = x_2 = 1$

$$X_2 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \text{ for } x_3 = 0$$

$$X_3 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \text{ for } x_3 = 1$$