CONVERGENCE OF JACOBI ITERATIONS

http://www.phys.uconn.edu/~rozman/Courses/m3511_19s/

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1 Jacobi iterations

Consider the solution of the following system of linear equations:

$$A\mathbf{x} = \mathbf{b}, \tag{1}$$

where A is a square *diagonally dominant* $n \times n$ matrix, and $\mathbf{x}$ and $\mathbf{b}$ are column vectors – $\mathbf{b}$ is given and $\mathbf{x}$ is yet unknown solution of the system of the equations:

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}, \qquad \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}, \qquad \mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}. \tag{2}$$

A square matrix is said to be diagonally dominant if for every row of the matrix, the magnitude of the diagonal element in that row is larger than or equal to the sum of the magnitudes of all non-diagonal elements in that row.

$$|a_{kk}| \geq \sum_{\substack{i=1 \\ i \neq k}}^{n} |a_{ki}|. \tag{3}$$

Let's split the matrix A into diagonal, D, and off diagonal, R parts:

$$A = D + R, \qquad D = \begin{bmatrix} a_{11} & 0 & \cdots & 0 \\ 0 & a_{22} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_{nn} \end{bmatrix}, \qquad R = \begin{bmatrix} 0 & a_{12} & \cdots & a_{1n} \\ a_{21} & 0 & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & 0 \end{bmatrix}. \tag{4}$$

Eq. (1) can be transformed as following:

$$(D + R)\mathbf{x} = \mathbf{b} \quad \rightarrow \quad D\mathbf{x} = \mathbf{b} - R\mathbf{x} \quad \rightarrow \quad \mathbf{x} = D^{-1}\mathbf{b} - D^{-1}R\mathbf{x}. \tag{5}$$

Let's introduce the notations

$$T = -D^{-1}R, \qquad \mathbf{c} = D^{-1}\mathbf{b}. \tag{6}$$

In this notations,

$$A = D + R = D(I - T). \tag{7}$$

We consider that last relation in Eq. (5) as the iteration process:

$$\mathbf{x}^{(k+1)} = T\mathbf{x}^{(k)} + \mathbf{c}, \qquad k = 0, 1, \ldots. \tag{8}$$

Here $\mathbf{x}^{(0)}$ is an arbitrary chosen vector and we expect that $\mathbf{x}^{(k)} \rightarrow \mathbf{x}$ as $k \rightarrow \infty$. The iterations Eq. (8) is called *Jacobi iterations*.

In matlab the Jacobi iterations can be coded, for example, as shown in Listing 1.

Consider the iterations in details:

$$\begin{aligned}
\mathbf{x}^{(1)} &= T\mathbf{x}^{(0)} + \mathbf{c} && (9)\\
\mathbf{x}^{(2)} &= T\left(T\mathbf{x}^{(0)} + \mathbf{c}\right) + \mathbf{c} = T^2\mathbf{x}^{(0)} + (I + T)\mathbf{c} && (10)\\
\mathbf{x}^{(3)} &= T\left(T^2\mathbf{x}^{(0)} + (I + T)\mathbf{c}\right) + \mathbf{c} = T^3\mathbf{x}^{(0)} + \left(I + T + T^2\right)\mathbf{c} && (11)\\
\cdots &\cdots \cdots \\
\mathbf{x}^{(k)} &= T^k\mathbf{x}^{(0)} + \left(I + T + T^2 + \ldots + T^{k-1}\right)\mathbf{c}. && (12)
\end{aligned}$$

Let's chose $\mathbf{c}$ as the initial approximation to the solution:

$$\mathbf{x}^{(0)} = c \tag{13}$$

Then,

$$\begin{aligned}\mathbf{x}^{(k)} &= \left(I + T + T^2 + T^3 + \ldots + T^k\right)\mathbf{c} && (14)\\ &= \left(\sum_{n=0}^{\infty} T^n\right)\mathbf{c} - T^{(k+1)}\left(\sum_{n=0}^{\infty} T^n\right)\mathbf{c} && (15)\end{aligned}$$

Assuming that the infinite sum

$$\sum_{n=0}^{\infty} T^n = I + T + T^2 + T^3 + \ldots \tag{16}$$

does exist, notice that

$$(I-T)\left(\sum_{n=0}^{\infty} T^n\right) = \left(\sum_{n=0}^{\infty} T^n\right)(I-T) = I. \tag{17}$$

Therefore,

$$\sum_{n=0}^{\infty} T^n = (I-T)^{-1}. \tag{18}$$

and

$$\mathbf{x}^{(k)} = (I-T)^{-1}\,\mathbf{c} - T^{(k+1)}\,(I-T)^{-1}\,\mathbf{c}. \tag{19}$$

The first term in Eq. (19)

$$\begin{aligned}(I-T)^{-1}\,\mathbf{c} &= \left(I + D^{-1}R\right)^{-1} D^{-1}\mathbf{b} = \left[D^{-1}(D+R)\right]^{-1} D^{-1}\mathbf{b} && (20)\\ &= \left[D^{-1}A\right]^{-1} D^{-1}\mathbf{b} = A^{-1}D\,D^{-1}\mathbf{b} = A^{-1}\mathbf{b} \equiv \mathbf{x} && (21)\end{aligned}$$

is the solution of Eq. (1):

$$(I-T)^{-1}\,\mathbf{c} = \mathbf{x} \tag{22}$$

Therefore the Jacobi iteration process can be written as

$$\mathbf{x}^{(k)} = \mathbf{x} - T^{(k+1)}\mathbf{x}. \tag{23}$$

The second term on the right in Eq. (23) is the error ofter k iterations.

The corresponding residue,

$$\mathbf{r}^{(k)} = \mathbf{b} - A\,\mathbf{x}^{(k)} = \mathbf{b} - A\left(\mathbf{x} - T^{(k+1)}\mathbf{x}\right) = A\,T^{(k+1)}\mathbf{x} = D\,(I-T)\,T^{(k+1)}\mathbf{x}. \tag{24}$$

Let λ_i and $\mathbf{e}_i$ be the eigenvalues and the corresponding eigenvectors of T:

$$T\,\mathbf{e}_i = \lambda_i \mathbf{e}_i, \quad i = 1, \ldots, n. \tag{25}$$

For every row of matrix T the sum of the magnitudes of all elements in that row is less than or equal to one. Thus, the eigenvalues of T have the following bounds:

$$|\lambda_i| < 1. \tag{26}$$

Let

$$\lambda_{\max} = \max(\{\lambda\}), \qquad T\,\mathbf{e}_{\max} = \lambda_{\max}\mathbf{e}_{\max}. \tag{27}$$

For $k \gg 1$

$$T^k\,\mathbf{x} \approx \alpha\lambda_{\max}^k \mathbf{e}_{\max}. \tag{28}$$

$$\left|\mathbf{r}^{(k)}\right|^2 = \lambda_{\max}^{2k}\alpha^2 |A\mathbf{e}_{\max}|^2 \sim \lambda_{\max}^{2k} = e^{2k\ln(\lambda_{\max})}. \tag{29}$$

```
function [x,  conv]=myjacobi(A, b, tol, maxit)
% MYJACOBI - solve Ax=b using Jacobi iterations
%            use c as the initial approximation for x.

%            Educational version - returns the solution
%            and the convergence information.
    [n, ~] = size(A);
    T = A;
    d = diag(A);
    for i = 1:n
         T(i,i) = 0;
    end
    c = b ./ d;
    T = T ./ d;
    x = c;
    r = A*x - b;
    delta = r'*r;
    conv = delta;
    delta0 = delta;
    iter = 0;
    while (iter < maxit) && (delta > delta0*tol)
        x = c - T*x;
        r = A*x - b;
        delta = r'*r;
        conv = [conv delta];
        iter = iter + 1;
    end
end
```

Listing 1: MATLAB implementation of Jacobi iterations