Name: ______________________________

Date: May 9, 2019

Question:	1	2	3	4	Total
Points:	25	40	25	10	100
Score:					

- Show all your work and indicate your reasoning in order to receive the credit.
- When analytic calculations are requested, submit them on separate pages. Present your answers in *low-entropy* form.
- Your matlab code must run, produce results, and must not print results of intermediate irrelevant calculations
- Name matlab script you write for Problem N part (x) as following: **fNx.m**. E.g. the script for Problem 6 part (d) should be named **f6d.m**
- Your are not allowed to collaborate with anyone

Instructor's comments:

1. (25 points) The following finite difference scheme

$$\frac{u_j^{n+1}-u_j^n}{\Delta t} = \frac{D}{2}\,\frac{u_{j+1}^{n+1}-2u_j^{n+1}+u_{j-1}^{n+1}}{\Delta x^2} + \frac{D}{2}\,\frac{u_{j+1}^{n}-2u_j^{n}+u_{j-1}^{n}}{\Delta x^2},$$

where $u_j^n = u(j\Delta x, n\Delta t)$, was developed in an attempt to solve numerically the partial differential equation,

$$\frac{\partial u}{\partial t} = D\frac{\partial^2 u}{\partial x^2},$$

where D is a positive constant, $u = u(x,t)$.

Use von Neumann stability analysis to show that the scheme above is unconditionally stable.

Hints: You may find the following formulas useful:

$$e^{i\theta} + e^{-i\theta} = 2\cos\theta,$$

$$1-\cos\theta = 2\sin^2\frac{\theta}{2}$$

$$\left|\frac{1-\phi}{1+\phi}\right| \le 1 \quad \text{for} \quad \phi \ge 0.$$

2. **Shape of an ideal stringed bow**

A longbow is constructed from a uniform thin elastic rod of length L and moment of inertia of the crossection I. The bow is stringed with opening angles of θ_0 at both ends.

The cartesian coordinates of a deformed thin elastic rod subjected to an axial loading are governed by the following equations:

$$\frac{\mathrm{d}x}{\mathrm{d}s} = \cos\theta, \quad \frac{\mathrm{d}y}{\mathrm{d}s} = \sin\theta, \quad x(0)=0, \quad y(0)=0, \qquad \text{(A)}$$

where s is the (relative) distance measured along the rod from its left end, and $\theta(s)$ is the tangential angle at point s (see Fig. 1). The equation that describes θ is as following

$$\frac{\mathrm{d}^2\theta}{\mathrm{d}s^2} + p\sin\theta = 0, \quad 0 \le s \le 1. \qquad \text{(B)}$$

For the reference, $p = \frac{PL^2}{EI}$, where P is the axial load, E is the Young's modulus, L is the length of the rod, and I is the area moment of inertia of the crossection of the rod.

For the stringed bow the boundary conditions for Eq. (B) are

$$\theta(0) = \theta_0, \quad \theta(1) = -\theta_0. \qquad \text{(C)}$$

Write the matlab code that solves the following problems. Use $p = 4\pi^2$.

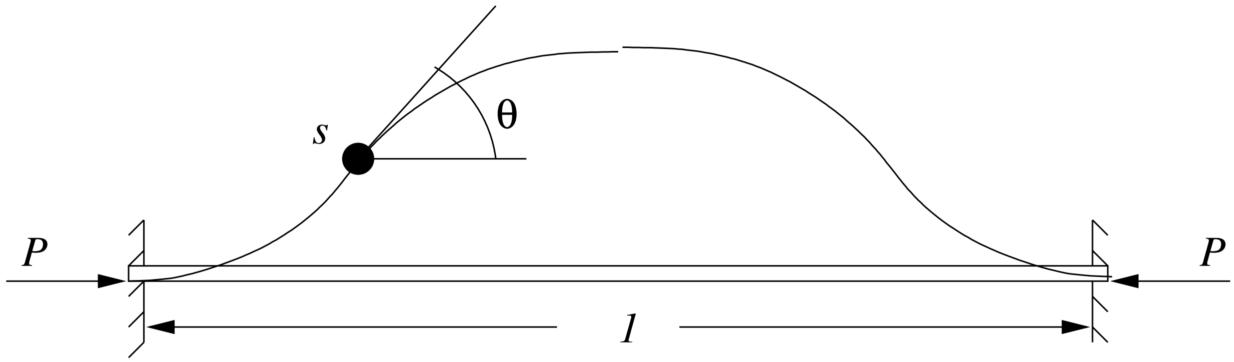

Figure 1: The sketch of the deformed rod geometry, the case of clamped ends.

(a) (5 points) Derive the equation for $z(s) = \dfrac{\partial \theta}{\partial t}$, where $t \equiv \theta'(0)$.

(b) (10 points) Use the Nonlinear Shooting with Newton's method to approximate the solution to the *boundary value problem* Eq. (B) and (C). For the Newton's method use `tol = 0.000001` as the error tolerance, and 5 as the intial approximation for $\theta'(0)$.

Plot the solution $\theta(s)$ for $\theta_0 = 0$.

(c) (10 points) Using the value of $\theta'(0)$ that you found in Part 2(b), solve the *initial value problem* for the Equations (A) and (B) and find $x(s)$, $y(s)$, $\theta(s)$.

Plot your numerical solution for the shape of the road $y(x)$. Print the displacement of the rod's right end due to the deformation. Print the largest lateral deformation of the rod, $\max(y(s))$.

(d) (15 points) For sufficient number of data points for θ_0 between 0.05 and 0.0001 calculate the maximal lateral deformation of the bow, $d(\theta_0)$. Plot the graph $d(\theta_0)$.

3. Find the numerical solution to the following elliptic partial differential equation:

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \left(x^2 + y^2\right) e^{xy}, \quad 0 < x < 1, \quad 0 < y < 1,$$

with the following boundary conditions:

$$u(0, y) = 1, \quad u(1, y) = e^y, \quad 0 \le y \le 1;$$

$$u(x, 0) = 1, \quad u(x, 1) = e^x, \quad 0 \le x \le 1.$$

Use finite difference approach.

(a) (5 points) Verify that the exact solution of the boundary value problem above is

$$u_e(x, y) = e^{xy}.$$

(b) (10 points) Write matlab code that solves the boundary value problem for PDE above. Plot your solution $u(x,y)$.

(c) (10 points) Compare your numerical result to the exact solution $u_e(x,y)$: plot the infinity norm of the error vs the mesh size. Describe your observations in your project's README file.

Git and Gitlab

4. (10 points) Upload **all** the code you wrote and you used for this exam:

   1. Use gitlab web interface to create a new project called **final** (the name is case sensitive, must be exactly as shown)
   2. Use gitlab web interface to create *README.md* file
   3. Use gitlab web interface to upload your matlab code to your project
   4. Use gitlab web interface to grant the access to your project (with the permission of the *reporter*) to the instructor.