Question:	1	2	3	4	Total
Points:	25	40	25	10	100

1. (25 points) The following finite difference scheme of first order forward difference in time and centered, second order differences in space,

$$\frac{u(x,t+\Delta t)-u(x,t)}{\Delta t}+\frac{-\frac{1}{2}u(x-2\Delta x,t)+u(x-\Delta x,t)-u(x+\Delta x,t)+\frac{1}{2}u(x+2\Delta x,t)}{\Delta x^3}=0,$$

was developed in an attempt to solve numerically the partial differential equation,

$$\frac{\partial u}{\partial t}+\frac{\partial^3 u}{\partial x^3}=0.$$

Use von Neumann stability analysis to show that the scheme above is unconditionally unstable (and thus should never be used).

2. The cartesian coordinates of a deformed thin elastic rod with clamped ends subjected to an axial loading are governed by the following equations:

$$\frac{\mathrm{d}x}{\mathrm{d}s}=\cos\theta,\quad \frac{\mathrm{d}y}{\mathrm{d}s}=\sin\theta,\quad x(0)=0,\quad y(0)=0, \qquad \text{(A)}$$

where s is the (relative) distance measured along the rod from its left end, and $\theta(s)$ is the tangential angle at point s (see Fig. 1),

$$\frac{\mathrm{d}^2\theta}{\mathrm{d}s^2}+p\sin\theta=0,\quad 0\le s\le 1. \qquad \text{(B)}$$

For the reference,

$$p=\frac{PL^2}{EI},$$

where P is the axial load, E is the Young's modulus, L is the length of the rod, and I is the area moment of inertia of the crossection of the rod. For example, for a steel rod ($E\approx 2\times 10^{11}$ N/m^2) of length $L=1$ m and of the square crossection of the area of 1 cm^2 ($I\sim 10^{-9}$ m^4), the load of $P=10^3$ N gives $p=5$.

For the clamped ends the boundary conditions for Eq. (B) are

$$\theta(0)=0,\quad \theta(1)=0. \qquad \text{(C)}$$

Write the matlab code that solves the following problems. Use $p=4\pi^2$.

(a) (5 points) Derive the equation for $w(s)=\dfrac{\partial\theta}{\partial t}$, where $t\equiv\theta'(0)$.

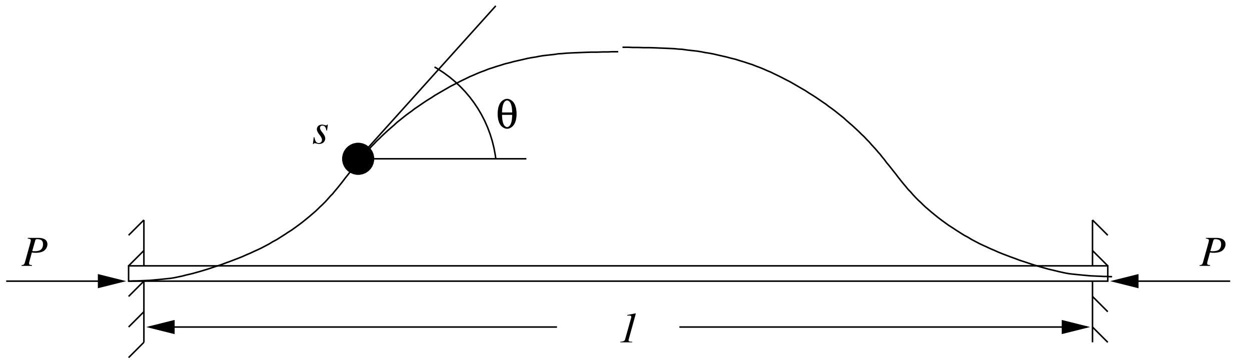

Figure 1: The sketch of the deformed rod geometry

(b) (20 points) Use the Nonlinear Shooting with Newton's method to approximate the solution to the *boundary value problem* Eq. (B) and (C). For the Newton's method use `tol = 0.000001` as the error tolerance, and `0.2` as the intial approximation for $\theta'(0)$.

Plot the solution $\theta(s)$.

(c) (15 points) Using the value of $\theta'(0)$ that you found in Part 2(b), solve the *initial value problem* for the Equations (A) and (B) and find $x(s)$, $y(s)$, $\theta(s)$.

Plot your numerical solution for the shape of the road $y(x)$. Print the displacement of the rod's right end due to the deformation. Print the largest lateral deformation of the rod, $\max(y(s))$.

3. Find the numerical solution to the following elliptic partial differential equation:

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = -\left(x^2 + y^2\right)\sin(xy), \quad 0 < x < \pi, \quad 0 < y < \pi,$$

with the following boundary conditions:

$$u(0, y) = 0, \quad u(\pi, y) = \sin(\pi y), \quad 0 \le y \le \pi;$$

$$u(x, 0) = 0, \quad u(x, \pi) = \sin(\pi x), \quad 0 \le x \le \pi.$$

Use finite difference approach.

(a) (5 points) Verify that the exact solution of the boundary value problem above is

$$u_e(x, y) = \sin(xy).$$

(b) (10 points) Write matlab code that solves the boundary value problem for PDE above. Plot your solution $u(x, y)$.

(c) (10 points) Compare your numerical result to the exact solution $u_e(x, y)$: plot the infinity norm of the error vs the mesh size. Describe your observations in your project's README file.

4. (10 points) Upload **all** the code you wrote and you used for this exam:

    1. Use gitlab web interface to create a new project called **final** (the name is case sensitive, must be exactly as shown)
    2. Use gitlab web interface to create *README.md* file
    3. Use gitlab web interface to upload your matlab code to your project
    4. Use gitlab web interface to grant the access to your project (with the permission of the *reporter*) to the instructor.