

WHAT IS THE CONDITION NUMBER OF A MATRIX?

<https://blogs.mathworks.com/cleve/2017/07/17/what-is-the-condition-number-of-a-matrix/>



Last modified: February 26, 2019

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1 Condition number

A condition number for a matrix measures how sensitive is the answer to our particular problem that involves the matrix to perturbations in the input data and to roundoff errors made during the solution process.

There are many different condition numbers. A condition number applies not only to a particular matrix, but also to the problem being solved. Are we solving linear equations, inverting a matrix, finding its eigenvalues, or computing the exponential? A matrix can be poorly conditioned for inversion while the eigenvalue problem is well conditioned. Or, vice versa.

However, when we simply say a matrix is “ill-conditioned”, we are usually just thinking

of the sensitivity of its inverse, i.e. of *the condition number for inversion*, and not of all the other condition numbers.

2 Norms

In order to make the sensitivity notions more precise, let's start with a vector norm. Specifically, with the *Euclidean norm* or *2-norm*:

$$\|x\| \equiv \left(\sum_i x_i^2 \right)^{1/2}. \quad (1)$$

The corresponding norm of a matrix A measures how much the multiplication by that matrix can stretch vectors.

$$M = \|A\| \equiv \max \frac{\|Ax\|}{\|x\|}. \quad (2)$$

It is sometimes also important to consider how much a matrix can shrink vectors.

$$m = \min \frac{\|Ax\|}{\|x\|}. \quad (3)$$

The reciprocal of the minimum stretching is the norm of the inverse of the matrix. Indeed,

$$m = \min \frac{\|Ax\|}{\|x\|} = \min \frac{\|y\|}{\|A^{-1}y\|} = \frac{1}{\max \frac{\|A^{-1}y\|}{\|y\|}} = \frac{1}{\|A^{-1}\|}. \quad (4)$$

A *singular* matrix is one that can map nonzero vectors into the zero vector. For a singular matrix

$$m = 0, \quad (5)$$

and the inverse does not exist.

The ratio of the maximum to minimum stretching is the condition number for inversion.

$$\kappa(A) \equiv \frac{M}{m}. \quad (6)$$

An equivalent definition is

$$\kappa(A) = \|A\| \|A^{-1}\|. \quad (7)$$

If a matrix is singular, then its condition number is infinite. A finite large condition number means that the matrix is close to being singular.

3 Linear equations

The condition number $\kappa(A)$ is involved in the answer to the question: how much can a change in the right hand side of a system of simultaneous linear equations affect the solution? Consider a system of equations:

$$Ax = b, \quad (8)$$

and a second system obtained by altering the right-hand side.

$$A(x + \delta x) = b + \delta b. \quad (9)$$

Think of δb as being the error in b and δx as being the resulting error in x , although we need not make any assumptions that the errors are small. Because $A(\delta x) = \delta b$, the definitions of M and m immediately lead to

$$M\|x\| \geq \|b\|, \quad \|x\| \geq \frac{\|b\|}{M}, \quad (10)$$

and

$$m\|\delta x\| \leq \|\delta b\|, \quad \|\delta x\| \leq \frac{\|\delta b\|}{m}. \quad (11)$$

Consequently, if $m \neq 0$,

$$\frac{\|\delta x\|}{\|x\|} \leq \kappa(A) \frac{\|\delta b\|}{\|b\|}. \quad (12)$$

The quantity $\|\delta b\|/\|b\|$ is the *relative* change in the right-hand side, and the quantity $\|\delta x\|/\|x\|$ is the resulting *relative* change in the solution. The advantage of using relative changes is that they are *dimensionless* — they are not affected by overall scale factors.

This inequality shows that the condition number is a relative error magnification factor. Changes in the right-hand side can cause changes $\kappa(A)$ times as large in the solution.

4 Matlab example

Let's investigate a system of linear equations involving

$$A = \begin{pmatrix} 4.1 & 2.8 \\ 9.7 & 6.6 \end{pmatrix}. \quad (13)$$

Take b to be the first column of A , so the solution to $Ax = b$ is simply

$$x = \begin{pmatrix} 1 \\ 0 \end{pmatrix}. \quad (14)$$

In matlab:

```
>> A = [4.1 2.8; 9.7 6.6]
```

```
A =  
    4.1000    2.8000  
    9.7000    6.6000
```

```
>> b = A(:,1)
```

```
b =  
    4.1000  
    9.7000
```

```
>> x = A\b
```

```
x =  
    1  
    0
```

Now add 0.01 to the first component of b .

```
>> b2 = [4.11; 9.7]
```

```
b2 =  
    4.1100  
    9.7000
```

The solution changes dramatically.

```
>> x2 = A\b2
```

```
x2 =  
    0.3400  
    0.9700
```

This sensitivity of the solution x to changes in the right hand side b is a reflection of a large value of the condition number.

```
>> kappa = cond(A)
```

```
kappa =  
    1.6230e+03
```

The matlab function `cond` calculates the condition number per definition Eq. (7). For large matrices the exact calculations can be computationally too expensive. Another matlab function, `condest`, estimate the condition number by approximating $\|A^{-1}\|$ without calculating A^{-1} .

```
>> kappaest = condest(A)
```

```
kappaest =  
    2.2494e+03
```

The *upper bound* on the possible change in x

```
>> kappa*norm(b-b2)/norm(b)
```

```
ans =  
    1.5412
```

The *actual change* in x resulting from this perturbation is

```
>> norm(x-x2)/norm(x)
```

```
ans =  
    1.1732
```

So this particular change in the right hand side generated almost the largest possible change in the solution.