# Midterm I sample solutions

1. (a) $x = (12, -5, 0)^t$

$$|x|_2 = \sqrt{12^2 + 5^2 + 0^2} = \boxed{13}$$

$$|x|_\infty = \max(12, |-5|, 0) = \boxed{12}$$

(b) $x = (1, 2, 1, 4)^t$

$$|x|_2 = \sqrt{1+4+1+16} = \boxed{\sqrt{22}}$$

$$|x|_\infty = \max(1, 2, 1, 4) = \boxed{4}$$

(c) $x = \left(\frac{a}{\sqrt{a^2+b^2}}, \frac{b}{\sqrt{a^2+b^2}}, 2\right)^t$

$$|x|_2 = \Big[\underbrace{\frac{a^2}{a^2+b^2} + \frac{b^2}{a^2+b^2}}_{1} + 4\Big]^{1/2} = \boxed{\sqrt{5}}$$

$$|x|_\infty = \max\Big(\underset{\underset{1}{\leq}}{\frac{|a|}{\sqrt{a^2+b^2}}}, \underset{\underset{1}{\leq}}{\frac{|b|}{\sqrt{a^2+b^2}}}, 2\Big) =$$

$$= \max(1, 1, 2) = \boxed{2}$$

2. (a) Sum of <u>absolute values</u> of matrix elements in each row;

$$\begin{bmatrix} 10 \\ 14 \\ 5 \\ 3 \\ 3 \\ 3 \\ 2 \\ 3 \end{bmatrix} \leftarrow \text{max of those is } \|\cdot\|_\infty$$

$$\boxed{\|A\|_\infty = 14}$$

(b) Identical to a HW problem

3. $I^3 = I \cdot \underbrace{I \cdot I}_{I} = I \cdot I = I$

$\|I\|_\infty = 1 \leftarrow$ homework

$\|I\|_2 = 1 \swarrow$

4. (a) $A = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$. Eigenvalues are the roots of the characteristic polynomial: $\det(A - \lambda) = \det\begin{pmatrix} 2-\lambda & -1 \\ -1 & 2-\lambda \end{pmatrix} = 0$

$(2-\lambda)^2 - 1 = \lambda^2 - 4\lambda + 3 = 0$; $\lambda_{1,2} = 2 \pm \sqrt{4-3}$

$\boxed{\lambda_1 = 1}$; $\boxed{\lambda_2 = 3}$

Spectral radius of A: $\max\{|\lambda_i|\} = 3$

Eigenvectors are the solutions of the following linear system:

$(A-\lambda)X=0;\qquad \begin{pmatrix} 2-\lambda & -1 \\ -1 & 2-\lambda \end{pmatrix}\begin{pmatrix} X_1 \\ X_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

For $\lambda=\lambda_1$:

$\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} X_1 \\ X_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix};\qquad X_1 - X_2 = 0$ - only one eq. needed

$X_1 = X_2 = \underline{1}$ ← arbitrary

$\bar{X}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$. Let's normalize the eigenvector.

$|\bar{X}_1|_2 = \sqrt{1+1} = \sqrt{2};\qquad \boxed{\tilde{X}_1 = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix}};\quad |\tilde{X}_1| = \underline{1}$

For $\lambda=\lambda_2$:

$\begin{pmatrix} -1 & -1 \\ -1 & -1 \end{pmatrix}\begin{pmatrix} X_1 \\ X_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}\qquad X_1 = -X_2$ - only one eq. needed

$X_1 = -X_2 = 1$ ← arbitrary

$\bar{X}_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix};\quad |\bar{X}_2|_2 = \sqrt{2};$

$\boxed{\tilde{X}_2 = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix}}\qquad |\tilde{X}_2| = 1$

4(b) – identical method to 4(a)

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5. Condition number of a matrix A w.r.t. $|\cdots|_\infty$:

$$\varkappa(A) = |A|_\infty \cdot |A^{-1}|_\infty$$

(a) $A = \begin{pmatrix} 1 & -1 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & -1 \end{pmatrix}$; $|A|_\infty = 3$

Using matlab (inv(A)):

$A^{-1} = \begin{pmatrix} 1 & 1 & -2 \\ 0 & 1 & -1 \\ 0 & 0 & -1 \end{pmatrix}$;

Then, $|A^{-1}|_\infty = \max(4, 2, 1) = 4$

Finally, $\boxed{\varkappa(A) = 3 \cdot 4 = 12}$

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5(b) and (c) are similar to 5(a).

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Matlab code for myinversepower(...) has been discussed in class.