1 Tridiagonalization

Transforming a real symmetric matrix into a tridiagonal form
Given a real symmetrix $n \times n$ matrix A, want to find orthogonal matrices $P_1, P_2, ..., P_{n-2}$ such that

$$\underbrace{P_{n-2}P_{n-1}...P_2P_1}_{E}\, A \,\underbrace{P_1^T P_2^T ... P_{n-2}^T}_{E^T = E^{-1}} = V \text{ tridiagonal}$$

Note: The matrix P_k is designed to target the kth column of A, while P_k^T operates on the kth row of A.
Writing A and P_1 respectively as

$$A = \left(\begin{array}{c|c} a_{11} & a_1^T \\ \hline a_1 & A_1 \end{array}\right), \quad P_1 = \left(\begin{array}{c|c} 1 & 0^T \\ \hline 0 & H_1 \end{array}\right)$$

$$P_1AP_1^T = \left(\begin{array}{c|c} a_{11} & (H_1a_1)^T \\ \hline H_1a_1 & H_1A_1H_1^T \end{array}\right)$$

If we have

$$H_1a_1 = -\alpha_1 \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

then

$$P_1AP_1^T = \left(\begin{array}{c|c|c} a_{11} & -\alpha_1 & 0^T \\ -\alpha_1 & a_{22}^{(1)} & (a_2^{(1)})^T \\ \hline 0 & a_2^{(1)} & A_2 \end{array}\right)$$

Next set

$$P_2 = \left(\begin{array}{cc|c} 1 & 0 & 0^T \\ 0 & 1 & 0^T \\ \hline 0 & 0 & H_2 \end{array}\right)$$

then

$$P_2P_1AP_1^TP_2^T = \left(\begin{array}{cc|c} a_{11} & -\alpha_1 & 0^T \\ -\alpha_1 & a_{22}^{(1)} & (H_2a_2^{(1)})^T \\ \hline 0 & H_2a_2^{(1)} & H_2A_2H_2^T \end{array}\right)$$

Likewise, we want

$$H_2a_2^{(1)} = -\alpha_2 \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

Repeating the process $n-2$ times will yield a symmetric tridiagonal matrix.

1.1 Householder Transformation

Definition:
Given a vector u with unit length, the matrix

$$H = I - 2uu^T$$

is a Householder transformation.
Properties:

- H is symmetric
- H is orthogonal
- $||Hx|| = ||x||$ for any vector x

Remarks:

- Alternate form:
$$H = I - 2\frac{vv^T}{v^T v}$$
for any nonzero vector v.
- The $n \times n$ matrix H has $n-1$ free parameters
- It is not necessary to know H explicitly in order to compute Hy for any given vector y. [Only require u and $y^T u$.]

For a given vector x, want to find a vector u and a constant α such that

- $H = I - 2uu^T$ is a Housholder transformation
- The matrix H transforms x to a multiple of e_1, the first column of the identity matrix, i.e.
$$Hx = -\alpha e_1$$

Derivation:
Let $H = I - 2uu^T$ with $||u|| = 1$ and $Hx = -\alpha e_1$, i.e.

$$Hx = x - 2(u^T x)u = -\alpha e_1$$

Since H is an orthogonal matrix,

$$||x|| = ||Hx|| = |\alpha|$$

and so

$$\alpha = \pm ||x||$$

Also

$$x^H x = ||x||^2 - 2(u^T x)^2 = -\alpha e_1^T x$$

so

$$u^T x = \sqrt{||x||^2 \pm ||x|| e_1^T x}$$

To avoid catastrophic cancellation, set

$$\alpha = \text{sign } (e_1^T x)||x||$$

and thus

$$u^T x = \sqrt{\frac{1}{2}||x||(||x|| + |x_1|)}$$

$$u = \frac{x + \alpha e_1}{2(u^T x)}$$

Ex. 1) Let

$$A = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$$

To transform A to an upper Hessengberg form:

$$\text{Let } x = \begin{bmatrix} 1 \\ -1 \end{bmatrix}, ||x|| = \sqrt{2}, \text{ and } \alpha = +\sqrt{2}$$

$$2(u^T x)^2 = 2 + \sqrt{2} \Longrightarrow u^T x = \sqrt{1 + \frac{1}{\sqrt{2}}} = 1.30656296487638$$

$$u = \frac{1}{2\sqrt{1 + \frac{1}{\sqrt{2}}}} \left(\begin{bmatrix} 1 \\ -1 \end{bmatrix} + \sqrt{2} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right) = \begin{bmatrix} 0.92387953251129 \\ -0.38268343236509 \end{bmatrix}$$

Consequently, the Householder transformation is

$$H = \frac{1}{\sqrt{2}} \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix}$$

and $Hx = -\sqrt{2}e_1$.

Ex. 2) To transform $x = (-3, 1, 5, 1)^T$ to a multiple of the first column of the identity matrix,

$$||x|| = 6, \alpha = -6, u^T x = \sqrt{\frac{6^2 + 6 * 3}{2}} = 5.19615242270663$$

$$u = \frac{1}{2 \times 5.19615242270663} \left(\begin{bmatrix} -3 \\ 1 \\ 5 \\ 1 \end{bmatrix} - 6 \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \right) = \begin{bmatrix} -0.86602540378444 \\ 0.09622504486494 \\ 0.48112522432469 \\ 0.09622504486494 \end{bmatrix}$$

Consequently, the Householder transformation is

$$H = \frac{1}{54} \begin{bmatrix} -27 & -9 & -45 & -9 \\ -9 & 53 & -5 & -1 \\ -45 & -5 & -29 & -5 \\ -9 & -1 & -5 & 53 \end{bmatrix}$$

and $Hx = +6e_1$.