WHAT IS THE CONDITION NUMBER OF A MATRIX?

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1 Condition number for inversion

A condition number for a matrix measures how sensitive the answer is to perturbations in the input data and to roundoff errors made during the solution process.

I should point out that there are many different condition numbers. In general, a condition number applies not only to a particular matrix, but also to the problem being solved. Are we solving linear equations, inverting a matrix, finding its eigenvalues, or computing the exponential? A matrix can be poorly conditioned for inversion while the eigenvalue problem is well conditioned. Or, vice versa.

However, when we simply say a matrix is "ill-conditioned", we are usually just thinking of the sensitivity of its inverse, i.e. of *the condition number for inversion*, and not of all the other condition numbers.

2 Norms

In order to make the sensitivity notions more precise, let's start with a vector norm. Specifically, with the *Euclidean norm* or 2-*norm*:

$$\|x\| \equiv \left(\sum_i x_i^2 \right)^{1/2}. \tag{1}$$

The corresponding norm of a matrix A measures how much the mapping induced by that matrix can stretch vectors.

$$M = \|A\| \equiv \max \frac{\|Ax\|}{\|x\|}. \tag{2}$$

It is sometimes also important to consider how much a matrix can shrink vectors.

$$m = \min \frac{\|Ax\|}{\|x\|}. \tag{3}$$

The reciprocal of the minimum stretching is the norm of the inverse, because

$$m = \min \frac{\|Ax\|}{\|x\|} = \min \frac{\|y\|}{\|A^{-1}y\|} = \frac{1}{\max \frac{\|A^{-1}y\|}{\|y\|}} = \frac{1}{\|A^{-1}\|}. \tag{4}$$

A *singular* matrix is one that can map nonzero vectors into the zero vector. For a singular matrix

$$m = 0, \tag{5}$$

and the inverse does not exist.

The ratio of the maximum to minimum stretching is the condition number for inversion.

$$\kappa(A) \equiv \frac{M}{m}. \tag{6}$$

An equivalent definition is

$$\kappa(A) = \|A\| \, \|A^{-1}\|. \tag{7}$$

If a matrix is singular, then its condition number is infinite. A finite large condition number means that the matrix is close to being singular.

3 Linear equations

The condition number $\kappa(A)$ is involved in the answer to the question: how much can a change in the right hand side of a system of simultaneous linear equations affect the solution? Consider a system of equations:

$$Ax = b, \tag{8}$$

and a second system obtained by altering the right-hand side.

$$A(x + \delta x) = b + \delta b. \tag{9}$$

Think of δb as being the error in b and δx as being the resulting error in x, although we need not make any assumptions that the errors are small. Because $A(\delta x) = \delta b$, the definitions of M and m immediately lead to

$$\|b\| \leq M\|x\|, \tag{10}$$

and

$$\|\delta b\| \geq m\|\delta x\|. \tag{11}$$

Consequently, if $m \neq 0$,

$$\frac{\|\delta x\|}{\|x\|} \leq \kappa(A)\frac{\|\delta b\|}{\|b\|}. \tag{12}$$

The quantity $\|\delta b\|/\|b\|$ is the *relative* change in the right-hand side, and the quantity $\|\delta x\|/\|x\|$ is the resulting *relative* change in the solution. The advantage of using relative changes is that they are *dimensionless* — they are not affected by overall scale factors.

This inequality shows that the condition number is a relative error magnification factor. Changes in the right-hand side can cause changes $\kappa(A)$ times as large in the solution.

4 Matlab example

Let's investigate a system of linear equations involving

$$A = \begin{pmatrix} 4.1 & 2.8 \\ 9.7 & 6.6 \end{pmatrix}. \tag{13}$$

Take b to be the first column of A, so the solution to $Ax = b$ is simply

$$x = \begin{pmatrix} 1 \\ 0 \end{pmatrix}. \tag{14}$$

In matlab:

```
>> A = [4.1 2.8; 9.7 6.6]
```

```
A =
     4.1000     2.8000
     9.7000     6.6000
```

```
>> b = A(:,1)
```

```
b =
     4.1000
     9.7000
```

```
>> x = A\b
```

```
x =
     1
     0
```

Now add 0.01 to the first component of b.

```
>> b2 = [4.11; 9.7]
```

```
b2 =
     4.1100
     9.7000
```

The solution changes dramatically.

```
>> x2 = A\b2
```

```
x2 =
     0.3400
     0.9700
```

This sensitivity of the solution x to changes in the right hand side b is a reflection of a large value of the condition number.

```
>> kappa = cond(A)
```

```
kappa =
   1.6230e+03
```

The matlab function cond calculates the condition number per definition Eq. (7). For large matrices the exact calculations can be computationally too expensive. Another matlab function, condest, estimate the condition number by approximating $\|A^{-1}\|$ without calculating A^{-1}.

```
>> kappaest = condest(A)
```

```
kappaest =
   2.2494e+03
```

The *upper bound* on the possible change in x

```
>> kappa*norm(b-b2)/norm(b)
```

```
ans =
     1.5412
```

The value larger than one indicates changes in all of the significant digits.

The *actual change* in x resulting from this perturbation is

```
>> norm(x-x2)/norm(x)
```

```
ans =
     1.1732
```

So this particular change in the right hand side generated almost the largest possible change in the solution.

5 Condition number and inverse matrix

The condition number $\kappa(A)$ also appears in the bound for how much a change E in a matrix A can affect its inverse.

$$\frac{\|(A+E)^{-1}-A^{-1}\|}{\|A^{-1}\|} \le \kappa(A)\frac{\|E\|}{\|A\|} \tag{15}$$

Wilkinson's work about roundoff error in Gaussian elimination showed that each column of the computed inverse is a column of the exact inverse of a matrix within roundoff error of the given matrix. Let's fudge this a bit and say that inv(A) computes the exact inverse of $A+E$ where $\|E\|$ is on the order of roundoff error compared to $\|A\|$.

We don't know E exactly, but for an n-by-n matrix we have the estimate

$$norm(E) \approx n\text{eps}(\text{norm}(A)), \tag{16}$$

where matlab's `eps(x)` is the positive distance from `abs(x)` to the next larger in magnitude floating point number of the same precision as `x`.

So we have a simple estimate for the error in the computed inverse, relative to the unknown exact inverse. If `X = exact inverse of A` and `Z = inv(A)`, then

$$\frac{\|Z - X\|}{\|X\|} \approx n * \text{eps} * \text{cond}(A). \tag{17}$$

For our 2-by-2 example the estimate of the relative error in the computed inverse is

```
>> 2*eps*condest(A)
```

```
ans =
    9.9893e-13
```

This says we can expect 12 or 13 (out of 16) significant digits.

Wilkinson had to assume that every individual floating point arithmetic operation incurs the maximum roundoff error. But only a fraction of the operations have any roundoff error at all and even for those the errors are smaller than the maximum possible. So this estimate can be expected to an overestimate. But no tighter estimate is possible.

For our example, the computed inverse is

```
>> format long;
>> Z = inv(A)
```

```
Z =
 -66.000000000000242  28.000000000000103
  97.000000000000341 -41.000000000000142
```

It turns out that the exact inverse has the integer entries produced by

```
>> X = round(Z)
```

```
X =
 -66  28
  97 -41
```

We can compare the actual relative error with the estimate.

```
>> format short;
>> norm(Z - X)/norm(X)
```

```
ans =
     3.5583e-15
```

So we actually have about 15 significant digits of accuracy.