Question:	1	2	3	4	5	Total
Points:	15	40	10	20	15	100

Floating point numbers

1. Floating point numbers typically represented in computers in the following binary form:

$$\pm\left(1+\frac{b_1}{2}+\frac{b_2}{2^2}+\ldots+\frac{b_d}{2^d}\right)\times 2^E$$

   (a) (5 points) What is the (approximate) value of machine epsilon for a microprocessor that uses $d = 8$? Briefly explain.

   Machine epsilon is the separation between 1 and the next number that is larger than 1, i.e. $1 + 1/2^d \rightarrow \varepsilon = 2^{-d} = \boxed{2^{-8} = \frac{1}{256} \approx 0.004}$

   (b) (5 points) For the same microprocessor, how many floating point numbers x, such that $4 \le x < 5$ are there? Briefly explain.

   Floating point numbers are equidistant on the interval between 4 and 8. Hence, $N(4 \le x < 5) = \frac{1}{4} N(4 \le x < 8)$. $N(4 \le x < 8) = 2^d$

   $N(4 \le x < 5) = \frac{2^d}{4} = 2^{d-2} = 2^6 = \boxed{64}$

   (c) (5 points) For the same microprocessor, assuming that the smallest value of E is -16, what is (approximately) the smallest positive floating point number? Briefly explain.

   $X_{min} = 1 \cdot 2^{E_{min}} = 2^{-16} = \boxed{2^{-10} \cdot 2^{-6} \approx \frac{1}{1000 \cdot 50} \approx 2 \times 10^{-5}}$

Systems of linear equations

2. The chemical equation

$$x_1\,[Ca\,(OH)_2] + x_2\,[H\,N\,O_3] \rightarrow x_3\,[Ca\,(N\,O_3)_2] + 2\,[H_2O]$$

   indicates that x_1 molecules of calcium hydroxide $Ca\,(OH)_2$ combine with x_2 molecules of nitric acid $H\,N\,O_3$ to yield x_3 molecules of calcium nitrate $Ca\,(N\,O_3)_2$ and 2 molecules of water H_2O.

   Since atoms are not destroyed or created in chemical reactions, the balance of oxygen atoms requires that

$$2x_1 + 3x_2 = 6x_3 + 2.$$

The balance of hydrogen atoms requires that

$$2x_1 + x_2 = 4.$$

The balance for nitrogen atoms requires that

$$x_2 = 2x_3$$

(a) (5 points) Rewrite the balance equations above in matrix form $Ax = b$:

$$\begin{cases} 2x_1 + 3x_2 - 6x_3 = 2 \\ 2x_1 + x_2 = 4 \\ x_2 - 2x_3 = 0 \end{cases} \quad A = \begin{pmatrix} 2 & 3 & -6 \\ 2 & 1 & 0 \\ 0 & 1 & -2 \end{pmatrix}; \; b = \begin{pmatrix} 2 \\ 4 \\ 0 \end{pmatrix}$$

(b) (15 points) Use gaussian elimination without pivoting to reduce the matrix A to upper triangular form. Present your calculations, step by step, in the space below. Clearly indicate multiplication factors that you use.

$$\begin{pmatrix} 2 & 3 & -6 \\ 2 & 1 & 0 \\ 0 & 1 & -2 \end{pmatrix} \quad l_{21} = \frac{a_{21}}{a_{11}} = 1 \quad \begin{pmatrix} 2 & 3 & -6 \\ 2-2 & 1-3 & 0-(-6) \\ 0 & 1 & -2 \end{pmatrix} = \begin{pmatrix} 2 & 3 & -6 \\ 0 & -2 & 6 \\ 0 & 1 & -2 \end{pmatrix}$$

$$a_{31} = 0 \Rightarrow l_{31} = 0; \quad l_{32} = \frac{a_{32}}{a_{22}} = \frac{1}{-2} = -\frac{1}{2}$$

$$\begin{pmatrix} 2 & 3 & -6 \\ 0 & -2 & 6 \\ 0 & 1-(-2)\times(-\frac{1}{2}) & -2-6\cdot(-\frac{1}{2}) \end{pmatrix} = \begin{pmatrix} 2 & 3 & -6 \\ 0 & -2 & 6 \\ 0 & 0 & 1 \end{pmatrix}$$

(c) (5 points) Using the results of your gaussian elimination process write the lower triangular matrix L and the upper triangular matrix U such that $A = L \cdot U$.

$$L = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & -1/2 & 1 \end{pmatrix}; \quad U = \begin{pmatrix} 2 & 3 & -6 \\ 0 & -2 & 6 \\ 0 & 0 & 1 \end{pmatrix}$$

(d) (5 points) Use L and U to calculate the determinant of matrix A. Write you calculations below:

$$\det(A) = \det(L \cdot U) = \overbrace{\det(L)}^{1} \cdot \det(U) =$$

$$= U_{11} \cdot U_{22} \cdot U_{33} = 2 \cdot (-2) \cdot 1 = -4$$

(e) (5 points) Use the forward substitution to solve the equation $Ly = b$. Write you calculations below:

$$\begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & -1/2 & 1 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} 2 \\ 4 \\ 0 \end{pmatrix}; \quad \begin{array}{ll} y_1 = 2 & \\ y_1 + y_2 = 4 \rightarrow & y_2 = 2 \\ -\frac{1}{2} y_2 + y_3 = 0 & y_3 = 1 \end{array}$$

(f) (5 points) Use the backward substitution to solve the equation $Ux = y$. Verify by direct substitution that x is the solution of $Ax = b$. Write you calculations below:

$$\begin{pmatrix} 2 & 3 & -6 \\ 0 & -2 & 6 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix} \rightarrow \begin{array}{l} x_3 = 1 \\ -2x_2 + 6x_3 = 2 \rightarrow x_2 = 2 \\ 2x_1 + 3x_2 - 6x_3 = 2 \rightarrow \\ \rightarrow 2x_1 = 2 + 8 - 6 = 2; \ x_1 = 1 \end{array}$$

$$\boxed{x = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}} \quad \begin{pmatrix} 2 & 3 & -6 \\ 2 & 1 & 0 \\ 0 & 1 & -2 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 4 \\ 0 \end{pmatrix} \text{ - correct } b$$

3. (10 points) You wrote your own function to solve a system of linear equations. It takes about 10 seconds (on a slow computer) to solve the system of 100 equations with 100 unknowns. **Estimate** how long it would take to solve a system of 200 linear equations with 200 unknowns if your code implements LU-factorization method to solve the equations. Present your answer and explain your reasoning in the gitlab's README.md file.

Matlab

4. Write a script that measures the performance of matlab code. (Start the script with `clear clf` commands.)

   (a) (15 points) Preallocate a one dimensional array for storing your time measurements. Initialize the random number generator with a seed of your choice.

   For the size of matrix `n = 2000:100:3000` repeat the following steps:

   1. generate a random square matrix A of size $n \times n$ and a random column vector b of length n.
   2. factor the matrix A using matlab's builtin function `lu`.
   3. Solve the system of linear equations $Ax = b$ using the result of factorization and the functions `forwardsub` and `backwardsub` that we develop in class. Measure the solution time. Store the time into an element of the array you preallocated earlier.

   (b) (5 points) Plot the graph of the solution time vs. matrix size. Chose the type of the graph (linear or loglog). Plot the graph of the expected dependence $t(n)$. Provide labels, legent, title, grid.

Git and Gitlab

5. (15 points) Upload all the code you wrote/used for this exam:
   1. Use gitlab web interface to create a new project called **midterm1-sample** (the name is case sensitive, must be exactly as shown)
   2. Use gitlab web interface to add *README.md* file and edit it to add some meaningful content
   3. Use gitlab web interface to upload your matlab code to your project
   4. Use gitlab web interface to grant the access to your project (with the permission of the *reporter*) to the instructor.