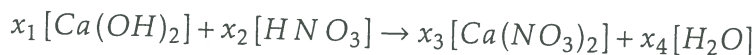


Question:	1	2	3	4	Total
Points:	45	10	20	15	90

Systems of linear equations

1. The chemical equation



indicates that x_1 molecules of calcium hydroxide $Ca(OH)_2$ combine with x_2 molecules of nitric acid HNO_3 to yield x_3 molecules of calcium nitrate $Ca(NO_3)_2$ and x_4 molecules of water H_2O .

Since atoms are not destroyed or created in chemical reactions, the balance of calcium atoms requires that

$$x_1 = x_3.$$

The balance of oxygen atoms requires that

$$2x_1 + 3x_2 = 6x_3 + x_4.$$

The balance of hydrogen atoms requires that

$$2x_1 + x_2 = 2x_4.$$

The balance for nitrogen atoms requires that

$$x_2 = 2x_3$$

(a) (5 points) Rewrite the balance equations above in matrix form $Ax = b$:

$$A = \begin{pmatrix} 1 & 0 & -1 & 0 \\ 2 & 3 & -6 & -1 \\ 2 & 1 & 0 & -2 \\ 0 & 1 & -2 & 0 \end{pmatrix}; \quad b = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

- (b) (5 points) Write matlab function (call it **chemreaction()**) that accepts no parameters and returns the 4×4 matrix A that you found in Step (a). When called, your function **must** print absolutely nothing. Provide help text that matlab will print if the command `help chemreaction` is typed.
- (c) (15 points) Use gaussian elimination without pivoting to reduce the matrix A to upper triangular form. Present your calculations, step by step, in the space below. Clearly indicate multiplication factors that you use.

First pass:

$$\begin{array}{l}
 1 \rightarrow \\
 2 \rightarrow \\
 3 \rightarrow \\
 4 \rightarrow
 \end{array}
 \left(\begin{array}{cccc}
 1 & 0 & -1 & 0 \\
 2 & 3 & -6 & -1 \\
 2 & 1 & 0 & -2 \\
 0 & 1 & -2 & 0
 \end{array} \right)$$

$$\text{mult}_{21} = 2$$

$$\text{mult}_{31} = 2$$

$$\text{mult}_{41} = 0$$

$$\begin{array}{r}
 \begin{array}{l}
 \text{2nd row} \\
 \text{1st row} \times 2 \\
 \hline
 \text{new 2nd row}
 \end{array}
 \begin{array}{rrrr}
 2 & 3 & -6 & -1 \\
 2 & 0 & -2 & 0 \\
 \hline
 0 & 3 & -4 & -1
 \end{array}
 \end{array}$$

$$\begin{array}{r}
 \begin{array}{l}
 \text{3rd row} \\
 \text{1st row} \times 2 \\
 \hline
 \text{new 3rd row}
 \end{array}
 \begin{array}{rrrr}
 2 & 1 & 0 & -2 \\
 2 & 0 & -2 & 0 \\
 \hline
 0 & 1 & 2 & -2
 \end{array}
 \end{array}$$

Second pass:

$$\begin{array}{l}
 1 \rightarrow \\
 2 \rightarrow \\
 3 \rightarrow \\
 4 \rightarrow
 \end{array}
 \left(\begin{array}{cccc}
 1 & 0 & -1 & 0 \\
 0 & 3 & -4 & -1 \\
 0 & 1 & 2 & -2 \\
 0 & 1 & -2 & 0
 \end{array} \right)$$

$$\text{mult}_{32} = \frac{1}{3}$$

$$\text{mult}_{42} = \frac{1}{3}$$

$$\begin{array}{r}
 \begin{array}{l}
 \text{3rd} \\
 \text{2nd row} \times \frac{1}{3} \\
 \hline
 \text{new 3rd row}
 \end{array}
 \begin{array}{rrrr}
 0 & 1 & 2 & -2 \\
 0 & 1 & -\frac{4}{3} & -\frac{1}{3} \\
 \hline
 0 & 0 & \frac{10}{3} & -\frac{5}{3}
 \end{array}
 \end{array}$$

$$\begin{array}{r}
 \begin{array}{l}
 \text{4th row} \\
 \text{2nd row} \times \frac{1}{3} \\
 \hline
 \text{new 4th row}
 \end{array}
 \begin{array}{rrrr}
 0 & 1 & -2 & 0 \\
 0 & 1 & -\frac{4}{3} & -\frac{1}{3} \\
 \hline
 0 & 0 & -\frac{2}{3} & \frac{1}{3}
 \end{array}
 \end{array}$$

Third pass:

$$\begin{array}{l}
 1 \rightarrow \\
 2 \rightarrow \\
 3 \rightarrow \\
 4 \rightarrow
 \end{array}
 \left(
 \begin{array}{cccc}
 1 & 0 & -1 & 0 \\
 0 & 3 & -4 & -1 \\
 0 & 0 & \frac{10}{3} & -\frac{1}{3} \\
 0 & 0 & \frac{2}{3} & -\frac{1}{3}
 \end{array}
 \right)$$

$\uparrow \quad \uparrow \quad \uparrow$
 $1 \quad 2 \quad 3$

$$\text{mult}_{43} = -\frac{1}{5}$$

$$\begin{array}{rcl}
 4\text{th row} & & 0 \quad 0 \quad -\frac{2}{3} \quad \frac{1}{3} \\
 3\text{rd row} \times (-\frac{1}{5}) & & 0 \quad 0 \quad -\frac{2}{3} \quad \frac{1}{3} \\
 \hline
 \text{new 4th row} & & 0 \quad 0 \quad 0 \quad 0
 \end{array}$$

Final
Result:

$$\left(
 \begin{array}{cccc}
 1 & 0 & -1 & 0 \\
 0 & 3 & -4 & -1 \\
 0 & 0 & \frac{10}{3} & -\frac{1}{3} \\
 0 & 0 & 0 & 0
 \end{array}
 \right)$$

- (d) (5 points) Using the results of your gaussian elimination process write the lower triangular matrix L and the upper triangular matrix U such that $A = L \cdot U$.

$$L = \begin{pmatrix} 1 & 0 & 0 & 0 \\ \text{mult}_{21} & 1 & 0 & 0 \\ \text{mult}_{31} & \text{mult}_{32} & 1 & 0 \\ \text{mult}_{41} & \text{mult}_{42} & \text{mult}_{43} & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ 2 & \frac{1}{3} & 1 & 0 \\ 0 & \frac{1}{3} & -\frac{1}{5} & 1 \end{pmatrix}$$

- (e) (5 points) Use L and U to calculate the determinant of matrix A . Write you calculations below:

$$U = \begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 3 & -4 & -1 \\ 0 & 0 & \frac{10}{3} & -\frac{5}{3} \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad \left| \quad \det(A) = \det(U) \right.$$

$$= 1 \cdot 3 \cdot \frac{10}{3} \cdot 0 = \boxed{0}$$

- (f) (5 points) Write a matlab script (call it chem.m) that prints the help for your function chemreaction(), calls your function to initialize A , creates the matrices L and U , and verifies the relation $A = L \cdot U$.
- (g) (5 points) Add some code to your script that verifies that the vector $[1; 2; 1; 2]$ is a solution of your system of equations.
2. (10 points) You wrote your own function to solve a system of linear equations. It takes about 10 seconds (on a slow computer) to solve the system of 100 equations with 100 unknowns. **Estimate** how long it would take to solve a system of 200 linear equations with 200 unknowns if your code implements gaussian elimination method. Present your answer and explain your reasoning in the gitlab's README.md file.

Matlab

3. Write a script that measures the performance of matlab code:

- (a) (15 points) Let's $n_{\min} = 100$ and $n_{\max} = 200$. Preallocate a one dimensional array for storing your time measurements. Initialize the random number generator.

For the size of matrix n , $n = n_{\min} : n_{\max}$ repeat the following steps:

1. generate a random square matrix A and a random column vector b of size n
2. warm up by solving the system of linear equations $A \cdot x = b$ using the function simplegauss that we developed in class

3. repeat the solution again measuring the time for the function call and store the time into an element of the array you preallocated earlier
- (b) (5 points) Plot the graph of time vs. matrix size. Chose the type of the axis (linear or log). Provide labels, title, grid. Use 'disconnected dots' line style.

Git and Gitlab

4. (15 points) Upload the code you wrote for this exam:
 1. Use gitlab web interface to create a new project called **midterm1-sample** (the name is case sensitive, must be exactly as shown)
 2. Use gitlab web interface to add *README.md* file and edit it to add some meaningful content
 3. Use gitlab web interface to upload your matlab code to your project
 4. Use gitlab web interface to grant the access to your project (with the permission of the *reporter*) to the instructor.