Show all your work and indicate your reasoning in order to receive the credit. Present your answers in *low-entropy* form.

Name: ______________________

Date: ______________________

Question:	1	2	3	4	Total
Points:	10	40	40	10	100
Score:					

1. (10 points) Find the general solution of the following equation:

$$(t-1)^2\frac{d^2y}{dt^2}+4(t-1)\frac{dy}{dt}+2y=0, \quad t>1.$$

Hint: introdice a new independent variable $x = t - 1$.

$x = t - 1 \Rightarrow \frac{dy}{dt} = \frac{dy}{dx}, \frac{d^2y}{dt^2} = \frac{d^2y}{dx^2}, x > 0$

The equation is

$$x^2\frac{d^2y}{dx^2} + 4x\frac{dy}{dx} + 2y = 0, \quad x > 0$$

This is Euler's equation; we look for its solution in the form: $y(x) = x^r$;

$\frac{dy}{dx} = r x^{r-1}$; $\frac{d^2y}{dx^2} = r(r-1)x^{r-2}$; $x\frac{dy}{dx} = r x^r$;

$x^2\frac{d^2y}{dx^2} = r(r-1)x^r$; The equation:

$r(r-1)\cancel{x^r} + 4r\cancel{x^r} + 2\cancel{x^r} = 0 \Rightarrow r^2 + 3r + 2 = 0$;

$r_{1,2} = -\frac{3}{2} \pm \sqrt{\frac{9}{4} - 2} = -\frac{3}{2} \pm \frac{1}{2}$; $r_1 = -2$, $r_2 = -1$;

General solution: $y = C_1 x^{r_1} + C_2 x^{r_2} = \boxed{\frac{C_1}{(t-1)^2} + \frac{C_2}{t-1}}$

2. Consider the following differential equation

$$\frac{d^2y}{dt^2} + t\frac{dy}{dt} + t^2y = 0.$$

Find the series solution about $t = 0$.

(a) (5 points) Classify the point $t = 0$ for the equation as ordinary, regular singular, or irregular singular.

(b) (10 points) Write down the solution as a series about $t = 0$ and, if applicable, determine the indicial equation and find the corresponding roots.

(c) (15 points) Find the recurrence relation that determines the coefficients in a series solution to this equation about $t = 0$.

(d) (10 points) Find the first four non-zero terms of the series solution of the following initial value problem for the equation above:

$$y(0) = 1, \quad y'(0) = 0.$$

(a) $p(t) = t$; $q(t) = t^2 \rightarrow p(0)=0$, $q(0)=0 \rightarrow$ $t=0$ is an $\boxed{\text{ordinary}}$ point of the equation.

(b) Thus the solution about $t=0$ has the form of Taylor series:

$$y(t) = \sum_{n=0}^{\infty} a_n t^n \qquad (*)$$

The concept of 'indicial' equation is not relevant here.

- $$t^2 y = \sum_{n=0}^{\infty} a_n t^{n+2} \underset{\substack{m=n+2 \\ n=m-2 \\ m=2,\dots}}{=} \sum_{m=2}^{\infty} a_{m-2} t^m \underset{(m\to n)}{=} \sum_{n=2}^{\infty} a_{n-2} t^n \qquad (**)$$

- $$y'(t) = \sum_{n=1}^{\infty} n a_n t^{n-1}; \quad ty' = \sum_{n=1}^{\infty} n a_n t^n \qquad (***)$$

Solution of Problem 2 continued:

- $y''(t) = \sum_{n=2}^{\infty} n(n-1) a_n t^{n-2} = \sum_{m=0}^{\infty} (m+2)(m+1) a_{m+2} t^m$ $\left(\begin{array}{l} n-2=m \\ n=m+2 \\ m=0,\dots \end{array}\right)$

$\underset{(m\to n)}{=} \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} t^n$ $(***x)$

Plug $(**)$, $(***)$, and $(***x)$ into the equation:

$$\underbrace{\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}t^n}_{a_2 + 6a_3 t + \sum_{n=2}^{\infty}(n+2)(n+1)a_{n+2}t^n} + \underbrace{\sum_{n=1}^{\infty} n a_n t^n}_{a_1 t + \sum_{n=2}^{\infty} n a_n t^n} + \sum_{n=2} a_{n-2} t^n = 0$$

$$a_2 + (6a_3 + a_1)t + \sum_{n=2}^{\infty} t^n \left[(n+2)(n+1)a_{n+2} + n a_n + a_{n-2}\right] = 0$$

(c) Thus $a_2 = 0$, $a_3 = -\frac{a_1}{6}$, $\boxed{a_{n+2} = -\frac{n a_n + a_{n-2}}{(n+2)(n+1)}}$

(d) From $(*)$: $y(0) = \boxed{a_0 = 1}$; $y'(0) = \boxed{a_1 = 0}$

$a_1 = 0 \to a_3 = -\frac{a_1}{6} = 0 \to a_5 = 0 \to \boxed{a_{2k+1} = 0}$

$a_0 = 1, a_2 = 0 \to a_4 = -\frac{1}{4\cdot 3} = -\frac{1}{12}$; $a_6 = -\frac{-1/12 + 0}{6\cdot 5} = \frac{1}{90}$;

$y(t) = 1 - \frac{1}{12}t^4 + \frac{1}{90}t^6 + \frac{1}{3360}t^8$ $a_8 = -\frac{6/90 + 1/12}{8\cdot 7} = \frac{1}{3360}$

3. Consider the following differential equation

$$t^2\frac{d^2y}{dt^2}+t\frac{dy}{dt}+(t-1)y=0.$$

Find the series solution about $t=0$.

(a) (5 points) Classify the point $t=0$ for the equation as ordinary, regular singular, or irregular singular.

(b) (10 points) Write down the solution in a series form and, if applicable, determine the indicial equation and find the corresponding roots.

(c) (15 points) Find the recurrence relation that determines the coefficients in a series solution.

(d) (10 points) Find the first four non-zero terms of the series solution of the following initial value problem:

$$y(0)=0, \quad y'(0)=1.$$

(a) $y''+\frac{1}{t}y'+\left(\frac{1}{t}-\frac{1}{t^2}\right)y=0 \to p(t)=\frac{1}{t}$, $q(t)=\frac{1}{t}-\frac{1}{t^2}$

as $t\to 0$ both $p(t)\to\infty$, $q(t)\to-\infty$ $\longrightarrow$ hence $t=0$ is a singular point of the equation.

Next, $t\cdot p(t)=1$, $t^2q(t)=t-1$ $\longrightarrow$ as $t\to 0$ both $tp(t)$ and $t^2q(t)$ are finite $\to$ thus $t=0$ is a regular singular point.

(b) Therefore, we can search for the solution in the form of Frobenius series: $y(t)=t^r\sum_{n=0}a_nt^n$, (*) where the values of r are determined by solving indicial equation.

(**) $y(t)=\sum_{n=0}^{\infty}a_nt^{n+r}$; $\quad ty(t)=\sum_{n=0}^{\infty}a_nt^{n+r+1}=\sum_{n=1}^{\infty}a_{n-1}t^{n+r}$ (***)

$y'(t)=\sum_{n=0}^{\infty}(n+r)a_nt^{n+r-1}$; $\quad ty'=\sum_{n=0}^{\infty}(n+r)a_nt^{n+r}$; (****)

$y''(t)=\sum_{n=0}^{\infty}(n+r)(n+r-1)a_nt^{n+r-2}$; $\quad t^2y''=\sum_{n=0}^{\infty}(n+r)(n+r-1)a_nt^{n+r}$ (*****)

Solution of Problem 3 continued: Plug (**), (***), (****), and (*****) into the equation.

$$\underbrace{\sum_{n=0}^{\infty}(n+r)(n+r-1)a_n t^{n+r}}_{r(r-1)a_0 t^r + \sum_{n=1}^{\infty}(n+r)(n+r-1)a_n t^{n+r}} + \underbrace{\sum_{n=0}^{\infty}(n+r)\,a_n t^{n+r}}_{r a_0 t^r + \sum_{n=1}^{\infty}(n+r)a_n t^{n+r}} - \underbrace{\sum_{n=0}^{\infty} a_n t^{n+r}}_{a_0 t^r + \sum_{n=1}^{\infty} a_n t^{n+r}} + \sum_{n=1}^{\infty} a_{n-1} t^{n+r} = 0$$

$$\underbrace{\left[r(r-1)+r-1\right]a_0 t^r}_{0} + \sum_{n=1}^{\infty} t^{n+r}\underbrace{\left\{\left[(n+r)(n+r-1)+(n+r)-1\right]a_n + a_{n-1}\right\}}_{0} = 0$$

Indicial equation: $r(r-1)+r-1=0 \to r^2=1 \to r_{1,2}=\pm 1$

(c) Recurrence relation: $(n+r-1)(n+r+1)a_n = -a_{n-1}$

$$a_n = -\frac{a_{n-1}}{(n+r)^2-1};$$

(d) $r_1 = 1$, $y_1 = a_0 t + a_1 t^2 + \ldots$; $r_2 = -1$, $y_2 = \frac{a_0}{t} + a_1 + \ldots$

Since $y(0)$ is finite, only $y_1(t)$ is relevant.

$y(t) = \sum_{n=0}^{\infty} a_n t^{n+1} \to y(0) = 0$; $\quad y' = \sum_{n=0} a_n t^n \cdot (n+1)$; $y'(0) = \boxed{a_0 = 1}$

$a_1 = -\frac{a_0}{4-1} = -\frac{1}{3}$; $\quad a_2 = -\frac{a_1}{(2+1)^2-1} = \frac{1}{24}$;

$a_3 = -\frac{a_2}{(3+1)^2-1} = -\frac{1}{24 \cdot 15} = -\frac{1}{360}$

$$\boxed{y(t) = t - \frac{t^2}{3} + \frac{t^3}{24} - \frac{t^4}{360} + \ldots}$$

4. Consider the differential equation

$$x^3 \frac{d^2y}{dx^2} + x\frac{dy}{dx} - y = 0.$$

(a) (5 points) Find and classify the finite singular point of the equation.

(b) (5 points) Find the Wronskian of the equation. Use the fact that $y_1(x) = x$ is a solution to find a second independent solution $y_2(x)$.

See HW 7, P2