Show all your work and indicate your reasoning in order to receive the credit. Present your answers in *low-entropy* form.

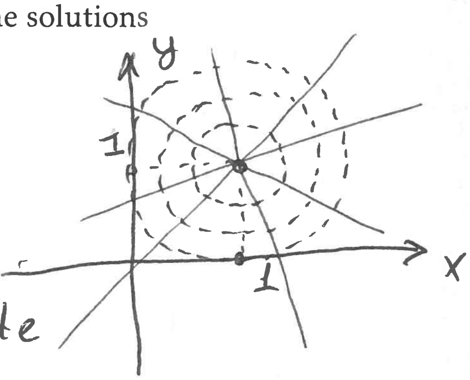
Name: ______________________

Date: 2/16/18

Question:	1	2	3	4	5	6	7	Total
Points:	10	10	15	5	10	10	15	75
Score:								

1. Find the orthogonal trajectories of the family of straight lines passing through the point $(1,1)$.
   (a) (3 points) Write the equation for the family of lines that contains a single parameter
   (b) (2 points) Obtain the differential equation for the family
   (c) (1 point) Write down the differential equation for the orthogonal family of curves
   (d) (4 points) Solve this differential equation and sketch the solutions

(a) $\boxed{(y-1) = C(x-1)}$ $(*)$



(b) Take derivative: $\frac{dy}{dx} = C$

we need to ~~exclude~~ elliminate the constant. From $(*)$:

$C = \frac{y-1}{x-1}$. Thus $\boxed{\frac{dy}{dx} = \frac{y-1}{x-1}}$

(c) Replace $\frac{dy}{dx} \to -\frac{dx}{dy}$; $\boxed{-\frac{dx}{dy} = \frac{y-1}{x-1}}$ $(**)$

(d) Solve $(**)$ separating variables: $-(x-1)d(x-1) = (y-1)d(y-1)$

$-\frac{(x-1)^2}{2} = \frac{(y-1)^2}{2} + c_1 \to \boxed{(x-1)^2 + (y-1)^2 = C^2}$ $(C^2 = -2c_1)$

The curves are circles withe the center at $(1,1)$

2. Consider the following autonomous equation:

$$\frac{dy}{dt} = (y-2)(y-1)^2.$$

(a) (1 point) Find all equilibrium solutions.

(b) (4 points) Classify the stability of each equilibrium solution.

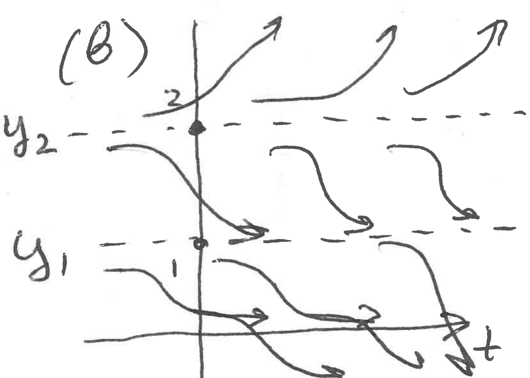
(c) (2 points) If $y(0) = 1.5$, what is limit $\lim_{t\to\infty} y(t)$?

(d) (3 points) If $y(-1) = 3$, what is limit $\lim_{t\to\infty} y(t)$?

(a) equilibrium sol: $\frac{dy}{dt} = 0 \to (y-2)(y-1)^2 = 0$

$\boxed{y_1 = 1}$ and $\boxed{y_2 = 2}$ are equilibrium solutions

(b)



$y > 2$ — $\frac{dy}{dt} > 0$ - sols are increasing

$1 < y < 2$ — $\frac{dy}{dt} < 0$ - sols. are decreasing

$y < 1$ — $\frac{dy}{dt} < 0$ — sols are decreasing

$\boxed{y_1 = 1 \text{ - semistable}, \quad y_2 = 2 \text{ — unstable}}$

(c) $y(0) = 1.5$ — the solution is trapped at all times and between $y=1$ and $y=2$. Hence $\boxed{\lim_{t\to\infty} y(t) = 1}$ it is decreasing.

(d) $y(-1) = 3$ — the solution is above $y = 2$ and it is increasing. Hence, $\boxed{\lim_{t\to\infty} y(t) = +\infty}$

3. (15 points) Use the method of reduction of order to find the general solution of the following differential equation:

$$(1-t^2)\frac{d^2y}{dt^2} - 2t\frac{dy}{dt} + 6y = 0. \qquad (*)$$

The known partial solution is

$$y_1(t) = 3t^2 - 1.$$

Canonical form: $\frac{d^2y}{dt^2} - \frac{2t}{1-t^2}\frac{dy}{dt} + \frac{6}{1-t^2}y = 0$

$p(t) = -\frac{2t}{1-t^2}$; $\int p(t)dt = \int \frac{d(t^2-1)}{t^2-1} = \ln|t^2-1|$

Wronskian: $W = Ce^{-\int p(t)dt} = Ce^{\ln\frac{1}{|t^2-1|}} = \boxed{\frac{C}{t^2-1}}$ later use $C=1$

The second solution is given by the expression:

$$y_2 = y_1\int \frac{W(t)}{y_1^2}dt = (3t^2-1)\int\frac{dt}{(t^2-1)(3t^2-1)^2}$$

Evaluating the integral (will be provided on the test if needed), obtain

$$\boxed{y_2 = 6t + (3t^2-1)\log\frac{(1-t)}{(1+t)}}$$

4. (5 points) Find the general solution of the following equation:

$$\frac{d^2y}{dt^2} - 7\frac{dy}{dt} + 12y = 0.$$

5. (10 points) Solve the initial value problem:

$$\frac{d^2y}{dt^2} - 6\frac{dy}{dt} + 9y = 0, \quad y(0) = 1, \quad y'(0) = 2.$$

#4 Characteristic equation: $r^2 - 7r + 12 = 0$

roots: $r_{1,2} = \frac{7}{2} \pm \sqrt{\frac{49}{4} - 12} = \frac{7}{2} \pm \frac{1}{2}$; $\boxed{r_1 = 4, \ r_2 = 3}$

General solution: $\boxed{y(t) = C_1 e^{4t} + C_2 e^{3t}}$

#5 Characteristic equation: $r^2 - 6r + 9 = 0$

One double root: $r = 3$

General solution: $y(t) = (C_1 + tC_2)e^{3t}$

Initial conditions: $y(0) = \boxed{1 = C_1}$

$y'(t) = C_2 e^{3t} + 3(C_1 + tC_2)e^{3t}$

$y'(0) = C_2 + 3C_1 = 2$ $\boxed{C_2 = 2 - 3 = -1}$

$\boxed{y = (1-t)e^{3t}}$

6. (10 points) Find the general solution of the equation

$$(1+t^2)\frac{dy}{dt} + ty = (1+t^2)^{\frac{5}{2}}. \quad *$$

Eq. (*) is first order linear ODE.

Canonical form: $\dfrac{dy}{dt} + \dfrac{t}{1+t^2}\, y = (1+t^2)^{3/2}$

Integrating factor: $\displaystyle\int \frac{t}{1+t^2}\,dt = \frac{1}{2}\int \frac{d(1+t^2)}{1+t^2} = \frac{1}{2}\ln(1+t^2)$

$\mu(t) = e^{\frac{1}{2}\ln(1+t^2)} = \boxed{(1+t^2)^{1/2}}$

$$\underbrace{(1+t^2)^{1/2}\frac{dy}{dt} + (1+t^2)^{-1/2}\, t\, y}_{\frac{d}{dt}\left((1+t^2)^{1/2} y\right)} = (1+t^2)^2 = 1 + 2t^2 + t^4$$

Integrating: $(1+t^2)^{1/2} y = \displaystyle\int (1+2t^2+t^4)\,dt + c$

$= t + \frac{2}{3}t^3 + \frac{1}{5}t^5 + c$

$$\boxed{y = \frac{t + \frac{2}{3}t^3 + \frac{1}{5}t^5 + c}{\sqrt{1+t^2}}}$$

7. (15 points) Find the general solution of the equation

$$e^{\frac{t}{y}}(y-t)\frac{dy}{dt} = -y\left(1+e^{\frac{t}{y}}\right). \qquad (*)$$

Hint:

$$\int \frac{v-1}{ve^{-\frac{1}{v}}+v^2}\,dv = \ln\left(1+ve^{\frac{1}{v}}\right)$$

The equation (*) is homogeneous first order ODE.

Indeed:

$$\frac{dy}{dt} = -\frac{y}{y-t}\left(e^{-\frac{t}{y}}+1\right) = \frac{\frac{y}{t}}{1-\frac{y}{t}}\left(e^{-\frac{1}{y/t}}+1\right)$$ depends upon y/t only.

Standard solution: $v=\frac{y}{t}$; $y=vt$; $\frac{dy}{dt}=v+t\frac{dv}{dt}$

$$v+t\frac{dv}{dt} = \frac{v}{1-v}\left(e^{-\frac{1}{v}}+1\right); \quad t\frac{dv}{dt} = \frac{ve^{-1/v}}{1-v}+\frac{v}{1-v}-v;$$

$$t\frac{dv}{dt} = \frac{ve^{-1/v}+\cancel{v}-\cancel{v}+v^2}{1-v} = \frac{ve^{-1/v}+v^2}{1-v}$$ – separable equation

$$\frac{(v-1)\,dv}{ve^{-1/v}+v^2} = -\frac{dt}{t}$$ – Integrating using the 'hint'

$$\ln\left(1+ve^{1/v}\right) = \ln\frac{1}{t}+c; \qquad 1+ve^{1/v}=\frac{C}{t}$$

Returning to y: $\boxed{1+\frac{y}{t}e^{t/y}=\frac{C}{t}}$