SPHERICAL SHELL UNDER PRESSURE

Spring semester 2026

https://www.phys.uconn.edu/~rozman/Courses/P3102_26S/

Last modified: February 19, 2026

$$\begin{cases} \sigma_{ij} = 2\mu\, u_{ij} + \lambda\, \delta_{ij} \left(\sum_k u_{kk} \right) \\ f_i + \sum_j \frac{\partial}{\partial x_j} \sigma_{ij} = 0 \end{cases} \tag{1}$$

In spherical polar coordinates r, θ, φ:

$$\mathbf{u} = (u_r(r),\ 0,\ 0), \tag{2}$$

$$\mathbf{f} = (0,\ 0,\ 0). \tag{3}$$

Navier-Cauchy equation of equilibrium:

$$(2\mu + \lambda) \frac{\mathrm{d}}{\mathrm{d}r} \left(\frac{1}{r^2} \frac{\mathrm{d}}{\mathrm{d}r} \left(r^2 u_r \right) \right) = 0. \tag{4}$$

Integrating Eq. (4):

$$\frac{1}{r^2} \frac{\mathrm{d}}{\mathrm{d}r} \left(r^2 u_r \right) = 3A, \quad \frac{\mathrm{d}}{\mathrm{d}r} \left(r^2 u_r \right) = 3Ar^2, \quad r^2 u_r = Ar^3 + B \tag{5}$$

$$u_r = Ar + \frac{B}{r^2}. \tag{6}$$

Elements of strain tensor for radial deformations in spherical coordinates:

$$u_{rr} = \frac{\partial u_r}{\partial r}, \quad u_{\theta\theta} = u_{\varphi\varphi} = \frac{u_r}{r}, \quad u_{\theta\varphi} = u_{r\theta} = u_{\varphi r} = 0. \tag{7}$$

For deformation Eq. (6),

$$u_{rr} = A - \frac{2B}{r^3}, \quad u_{\theta\theta} = u_{\varphi\varphi} = A + \frac{B}{r^3}, \quad u_{rr} + u_{\theta\theta} + u_{\varphi\varphi} = 3A. \tag{8}$$

Hooke's law:

$$\sigma_{rr} = 2\mu u_{rr} + \lambda\left(u_{rr} + u_{\theta\theta} + u_{\varphi\varphi}\right), \tag{9}$$

$$\sigma_{\theta\theta} = 2\mu u_{\theta\theta} + \lambda\left(u_{rr} + u_{\theta\theta} + u_{\varphi\varphi}\right), \tag{10}$$

$$\sigma_{\varphi\varphi} = 2\mu u_{\varphi\varphi} + \lambda\left(u_{rr} + u_{\theta\theta} + u_{\varphi\varphi}\right). \tag{11}$$

For the strain tensor Eq. (8),

$$\begin{aligned} \sigma_{rr} &= 2\mu\, u_{rr} + 3\lambda\left(u_{rr} + u_{\theta\theta} + u_{\varphi\varphi}\right) \\ &= 2\mu\left(A - \frac{2B}{r^3}\right) + 3\lambda A = (2\mu + 3\lambda)A - \frac{4B\mu}{r^3}, \end{aligned} \tag{12}$$

$$\begin{aligned} \sigma_{\theta\theta} &= 2\mu\, u_{\theta\theta} + 3\lambda\left(u_{rr} + u_{\theta\theta} + u_{\varphi\varphi}\right) \\ &= 2\mu\left(A + \frac{B}{r^3}\right) + 3\lambda A = (2\mu + 3\lambda)A + \frac{2B\mu}{r^3}, \end{aligned} \tag{13}$$

$$u_{\varphi\varphi} = \sigma_{\theta\theta} = (2\mu + 3\lambda)A + \frac{2B\mu}{r^3}. \tag{14}$$

Boundary conditions:

$$\sigma_{rr}(b) = 0, \quad \sigma_{rr}(a) = -p. \tag{15}$$

$$(2\mu + 3\lambda)A - \frac{4B\mu}{b^3} = 0, \tag{16}$$

$$(2\mu + 3\lambda)A - \frac{4B\mu}{a^3} = -p. \tag{17}$$

Subtracting Eq. (17) from (16),

$$4B\mu\left(\frac{1}{a^3} - \frac{1}{b^3}\right) = p, \tag{18}$$

$$B = \frac{p}{4\mu} \cdot \frac{1}{\frac{1}{a^3} - \frac{1}{b^3}} = \frac{p}{4\mu} \cdot \frac{a^3 b^3}{b^3 - a^3}. \tag{19}$$

From Eq. (16),

$$A = \frac{4B\mu}{(2\mu + 3\lambda)b^3} = \frac{p}{2\mu + 3\lambda} \cdot \frac{a^3}{b^3 - a^3}. \tag{20}$$

From Eq. (6)

$$u_r(r) = Ar + \frac{B}{r^2} = \frac{p}{2\mu + 3\lambda} \cdot \frac{a^3}{b^3 - a^3} r + \frac{p}{4\mu} \cdot \frac{a^3 b^3}{b^3 - a^3} \frac{1}{r^2} = r \frac{p}{4\mu} \frac{a^3}{b^3 - a^3}\left(\frac{4\mu}{2\mu + 3\lambda} + \frac{b^3}{r^3}\right). \tag{21}$$