CALCULATION OF THE SHIP WAVE PATTERN

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https://www.phys.uconn.edu/~rozman/Courses/P3102_26S/

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Figure 1: Examples of Kelvin's ship wave pattern

When a disturbance (e.g. a ship or a duck) travels on a water surface, it carries with it a pattern of waves known today as the Kelvin ship wake pattern.

The properties of a ship's wake follow entirely from the dispersion relation for deep-water gravity waves. In still water, a small-amplitude periodic disturbance of the surface with horizontal wave number $\mathbf{k} = k_x\hat{x} + k_y\hat{y}$ oscillates at angular frequency

$$\omega(\mathbf{k}) = \sqrt{g|\mathbf{k}|}, \tag{1}$$

where g is the acceleration of gravity. (Recall that the wavelength $\lambda = \frac{2\pi}{|\mathbf{k}|}$.)

The change due to waves in the water height at a point $\mathbf{x}$ at time t, $z(\mathbf{x}, t)$, can be written as a sum of contributions from all possible waves:

$$z(\mathbf{r}, t) = \int A(\mathbf{k})\, e^{i(\mathbf{k}\cdot\mathbf{r} - \omega(\mathbf{k})t)}\, \mathrm{d}^2\mathbf{k}, \tag{2}$$

for some unspecified yet amplitude $A(\mathbf{k})$.

We now place ourselves in the reference frame of a ship moving over the still water at velocity $\mathbf{u}$. In this frame, the water appears to be moving at velocity $-\mathbf{u}$, and the dispersion relations are Doppler shifted to

$$\omega(\mathbf{k}) = \sqrt{g|\mathbf{k}|} - \mathbf{u}\cdot\mathbf{k}. \tag{3}$$

Asking for stationary waves in the ship frame with $\omega(\mathbf{k}) = 0$, we reduce the wave vector integration in Eq. (2) to one dimensional integration (i.e. by the directional angle) by using the relation:

$$\mathbf{k} = \frac{g}{(\mathbf{u}\cdot\hat{\mathbf{k}})^2}\hat{\mathbf{k}}, \tag{4}$$

where $\hat{\mathbf{k}}$ is the unit vector in the direction of $\mathbf{k}$,

$$\hat{\mathbf{k}} = (\cos\theta, \sin\theta). \tag{5}$$

The angle θ is measured with respect to the direction of $\mathbf{u}$:

$$\mathbf{u}\cdot\hat{\mathbf{k}} = u\cos\theta. \tag{6}$$

The height is then given by

$$z(\mathbf{r}) = \int_0^{2\pi} A(\theta)\exp\left[ig\frac{\mathbf{r}\cdot\hat{\mathbf{k}}}{(\mathbf{u}\cdot\hat{\mathbf{k}})^2}\right]\mathrm{d}\theta. \tag{7}$$

The information about the ship is encoded in the amplitudes $A(\theta_k)$. Let assume that a point ship radiating backwards uniformly in all directions: A is constant over $\theta \in (-\pi/2, \pi/2)$, and zero outside it. Rewriting the integral in polar coordinates, r and ϕ,

$$\mathbf{r} = r(\cos\phi, -\sin\phi), \tag{8}$$

and introducing the dimensionless distance $\rho = \frac{g}{u^2}r$, we obtain

$$z(\phi,\rho) = \int_{-\pi/2}^{\pi/2} \cos\left(\rho\,\frac{\cos(\theta+\phi)}{\cos^2\theta}\right)\mathrm{d}\theta. \tag{9}$$

The wake pattern calculated using the integral Eq. (9) is presented in Fig. (2).

The section on *Integrals with Coalescing Saddles*, the Digital Library of Mathematical Functions (DLMF) [1] provides the analysis of the Kelvin wake pattern, starting from Equation (9).

References

[1] *NIST Digital Library of Mathematical Functions, §36.13 Kelvin's Ship-Wave Pattern*. F. W. J. Olver, et al. eds. URL: https://dlmf.nist.gov/36.13.

[2] Dana Longcope. *A ship's wake*. Department of Physics, Montana State University. 2015. URL: https://solar.physics.montana.edu/dana/ship_wake.pdf.

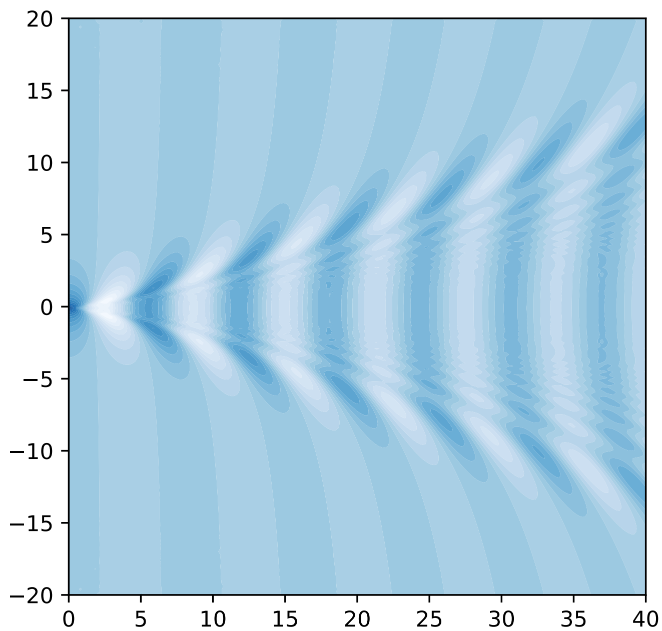

Figure 2: Ship wake calculated using Eq. (9). Coordinates are in units of $\frac{u^2}{g}$.