Elementary derivation of the wake pattern of a boat

Frank S. Crawford
Physics Department and Lawrence Berkeley Laboratory, University of California, Berkeley, California 94720

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An elementary derivation is given for the shape of the wake pattern of a boat traveling at constant velocity in deep water. We find essentially the same results as those found in standard treatments, but our mathematics is easier. All results follow from the fact that the group velocity is half the phase velocity.

I. INTRODUCTION

Most of us have seen the beautiful and complex wake pattern produced by a boat traveling at uniform velocity. (A photograph of a boat wake appears as Fig. 51-10 in Vol. 1 of Feynman's textbook.[1]) The pattern has two distinct components. (See Fig. 1.) One component consists of two wake lines that together form the arms of a "V" with the boat at the point. The waves along a wake line are "feathery" in appearance, because the individual waves do not propagate normal to the wake line but travel more forwards (in the direction of the boat velocity) than that. The individual featherlet waves also have a limited extension along their crests. The entire pattern moves as if it were attached to the boat. Remarkably, the pattern is nearly independent of boat speed, at sufficient speed (more than a few miles per hour). Each arm of the V makes an angle of about 19° with respect to the boat trajectory. Each featherlet wave crest makes an angle of about 55° with respect to the boat trajectory, and hence propagates at about 35° to the boat velocity.[2] The only dependence on boat speed is in the wavelength. The featherlet wavelength λ_0 is proportional to the square of the boat speed, and is 28 ft for a speed of 10 mph.

The second component of the wake pattern consists of transverse curved waves, each of which can be approximated by an arc of a circle. They fill the region between the arms of the V and cross the V so as to extend for a small distance outside the V. (In Fig. 1 for simplicity we show them as not crossing the V.) The radius of curvature R of each wave is equal to the distance L that the middle of the wave (the part that crosses the boat trajectory) lies behind the boat.[2] The center of the wave arc lies a distance $2R = 2L$ behind the boat. Where these waves cross the boat trajectory their wavelength λ_1 is 3/2 times the wavelength λ_0 of the featherlet waves (e.g., is 42 ft for a boat speed of 10 mph). This pattern is also "attached" to the boat.

This marvelously complex pattern follows entirely from the fact that these are dispersive waves for which the group velocity is half the phase velocity, which itself follows from the fact that phase velocity for a given wavelength is proportional to the square root of the wavelength. (We assume deep water and neglect surface tension which is only important for wavelengths much shorter than any we consider.)

An elegant and sophisticated derivation of this complex pattern was first given by Lord Kelvin.[3] Detailed derivations and discussions are found in well-known books.[4,5] These derivations are rigorous, but difficult. The following derivation is less rigorous than these, and easier, but gives essentially the same results.[6]

II. NONDISPERSIVE WAVES

We first review the simpler case of the wake (the shock wave front) produced by a point source of waves traveling uniformly in a straight line with a speed v_0 that exceeds the velocity v_ϕ of waves in a nondispersive medium. In such a medium all waves, of whatever wavelength, have the same phase velocity, and the group velocity equals the phase velocity. Such is the case for a bullet in air. In three dimensions the wake is a cone. In two dimensions it is a V-shaped pair of lines. By considering the wake to be the envelope of spherical waves emitted at successive locations of the point source, it is easy to see that the angle θ between the wave front—one of the arms of the V—and the velocity direction of the moving point source is given by[1]

$$v_\phi = v_0 \sin \phi, \tag{1}$$

as shown in Fig. 2. For the extreme case that the source velocity equals the phase velocity the V opens up into a straight wave propagating in the direction of the source and attached to the source. If the source velocity is less than the phase velocity there is no wake. Instead, there is a pattern of circular waves with the centers of the smaller circles closer to the latest boat position.

III. WATER WAVES

Now consider water waves. The dispersion relation is given by[7]

$$\omega^2 = gk, \tag{2}$$

where ω is the angular frequency, g is the gravitational acceleration, $k = 2\pi/\lambda$ is the wavenumber, and λ is the

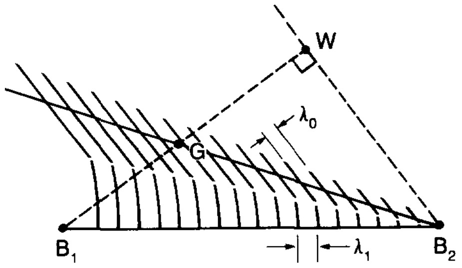

Fig. 1. Wake pattern of boat. (Only the "left" is shown.) The boat moves in the direction B_1 to B_2, where B_2 is the present boat position. Line B_2G is one of the two wake lines. (The other is symmetrically located on the other side of the boat line B_1B_2.) The wake line B_2G makes an angle of 19° with the boat line. The "featherlet" waves along the wake line propagate in the direction B_1W, which makes an angle of 35° with the boat line.

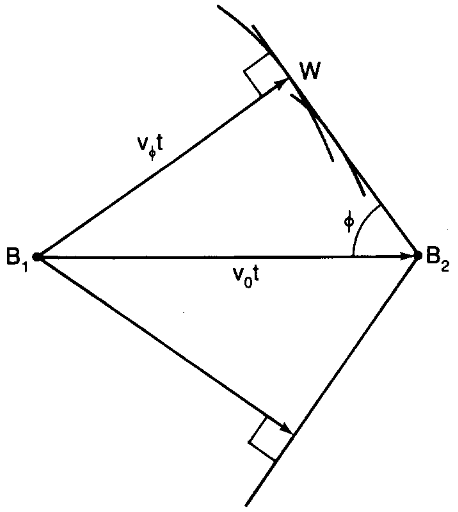

Fig. 2. Wake in nondispersive medium. The boat moves from B_1 to B_2 at velocity v_0 in time t. The wave front extends from B_2 through W when the boat is at B_2. The part of the wake near B_2 was emitted recently. The part near W was emitted when the boat was at B_1. It traveled to W at the phase velocity v_ϕ. Two of the circular waves of which B_2W gives the envelope are shown.

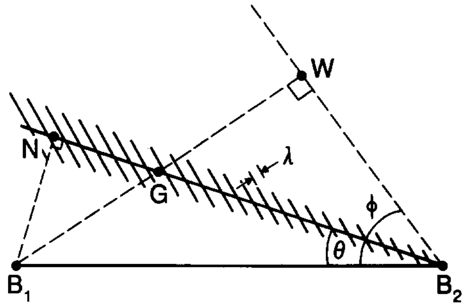

Fig. 3. Wake produced by narrow band of wavelengths centered at value λ. The phase velocity carries one crest of wavelength λ from B_1 to W while the boat goes from B_1 to B_2. Destructive interference within the band makes all the wave crest invisible except those near the wake line B_2G.

wavelength. The phase velocity v_ϕ is given by

$$v_\phi = \omega/k = (g/k)^{1/2} = (g\lambda/2\pi)^{1/2}; \tag{3}$$

i.e.,

$$\lambda = 2\pi v_\phi^2/g. \tag{4}$$

The group velocity is easily found by differentiating Eq. (2) to get

$$v_g = \frac{d\omega}{dk} = \frac{1}{2}\left(\frac{g\lambda}{2\pi}\right)^{1/2} = \frac{1}{2}v_\phi. \tag{5}$$

The entire wake pattern follows from Eq. (5).[8]

IV. THE V-SHAPED WAKE PRODUCED BY A NARROW BAND OF WAVELENGTHS

The point source traveling at velocity v_0 generates a succession of circular waves having a broad wavelength spectrum. Consider a given wavelength. Its phase velocity is given by Eq. (3). We consider only wavelengths with phase velocity less than the boat velocity, so that we get for this wavelength a V-shaped wake pattern like that of Fig. 1. Since we are considering a single wavelength the corresponding wave train must be very long. (For a precisely defined wavelength the wave train is in fact infinitely long.) Thus we have not only the wave crest that passes through the boat as in Fig. 2, but also a large number of crests, both ahead and behind that shown in Fig. 2. Since the boat is always generating an expanding circular wave right at the boat, there is always a crest that passes through the boat position and travels with the boat. Call this the "canonical" crest. It starts at B_2 and passes through W in Fig. 2. (We neglect the possibility of a phase shift that might cause the canonical crest to be displaced slightly with respect to the boat.)

Now consider a narrow band of wavelengths centered on our chosen wavelength. All the waves in the band propagate in nearly the same direction and with nearly the same phase velocity, *but not exactly*. Those with slightly longer or shorter wavelength travel slightly faster or slower. Consider the waves in this band that were generated when the boat was at B_1 of Fig. 2. The waves in the band started out in phase at B_1 when the boat was there. The waves of interest are all propagating towards point W. The canonical crest of our central wavelength arrives at W when the boat arrives at B_2. But the crests of slightly longer and shorter wavelengths arrive at W slightly sooner or later, and we therefore get destructive interference at W. Because the group velocity is half the phase velocity, and the group velocity is the velocity of the location where a narrow band of wavelengths continues to give constructive interference within the band, the major disturbance is half way between B_1 and W, not at W.

We can now construct the wake pattern for this narrow band of wavelengths. Construct Fig. 2 for any two boat positions B_1 and B_2, using the correct value of θ given by Eq. (1). Now make a point G (for group) half way between B_1 and W. Connect G and B_2 with a straight line. This is the wake line. Sketch in as "invisible" dotted lines all the wave crests of the central wavelength between W and B_1. In the region where a given crest crosses the wake line it becomes "visible." This is shown in Fig. 3. (The invisible parts of the wave crest are omitted.) Starting at any given part of the canonical crest B_2W one can travel back to the boat trajectory along the reverse of the crest propagation direction to find the point where that part of the canonical crest was generated. The crest halfway between the canonical crest and the generation point will be "visible," as will be several others slightly ahead or behind it.

V. FINDING THE DOMINANT WAKE ANGLE

Each narrow band of wavelengths will give a pattern similar to that of Fig. 3, but with a different wave crest angle ϕ and hence a different wake angle θ for each wavelength. We must decide which wavelengths are most important. This is easily done graphically, with no algebra, as follows. Draw the boat line B_1B_2. Now choose different values of ϕ. (These correspond to different phase velocities and hence different wavelengths.) For each value of ϕ constuct points W and G and draw the wake line B_2G. You will immediately notice the wake angle goes through a maximum for ϕ near 55°. In Fig. 4 we show this construction for $\phi = 40°$, 55°, and 70°, with corresponding values $\theta = 17°$, 19°, and 16°. Careful graphical construction (or the trigonometry given in the Appendix) shows this maximum to be at wake angle $\theta_0 = 19.5°$, with corresponding featherlet crest angle $\phi_0 = 54.7°$. According to Eq. (1) this corre-

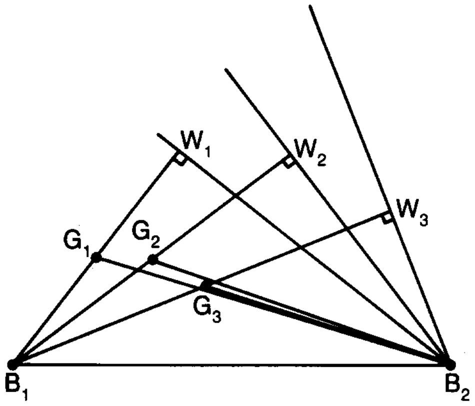

Fig. 4. Wake-angle construction for three different phase velocities. Phase velocities are proportional to B_1W, for $W = W_1$, W_2, and W_3. Wake angle B_1B_2G goes through a maximum at $G = G_2$.

sponds to a phase velocity of $0.82v_0$, and hence a group velocity $0.41v_0$. According to Eq. (4) that gives, for $v_0 = 10$ mph, a wavelength $\lambda_0 = 28$ ft, with λ_0 proportional to v_0^2.

In the wavelength region near λ_0 a wide band of wavelengths all give essentially the same wake angle of about 19°. They therefore dominate the wake and that is the wake we see. (This is sometimes called a Jacobian intensity maximum.) Figure 3 was drawn for this dominant wake angle.

The V-shaped wake pattern is attached to the boat, as is evident from the construction of Fig. 3. A surfer trying to ride a given featherlet crest forever must keep angling towards the boat to keep the crest from disappearing under her. In fact she must travel at exactly the boat velocity with respect to the water, to stay on the crest. Suppose instead the surfer tries to stay on a given crest of the dominant wavelength without traveling laterally along the crest. Start with the crest somewhere between B_1 and G. The surfing is difficult, since destructive interference is making the crest "invisible." But this crest is catching up with the wake line (the crest is traveling at twice the group velocity). As the surfer enters the wake region the wave amplitude starts to grow, and the surfer is at the right-hand end of a visible featherlet wave. As this crest moves through the wake its visible portion moves to the surfer's right. Finally the left end of the visible portion is reached and the wave amplitude again dies down to small values as the crest gets ahead of the wake. The surfing again becomes impossible. (The surfer can now wait for the wake to catch up with her, and try again.)

VI. THE TRANSVERSE WAKE

We now turn to the second component of the wake pattern, the transverse curved waves that follow the boat between the arms of the V. (See Fig. 1.) These waves are generated by point disturbances of the water by the boat as it moves along its trajectory. Any one of these disturbances produces a circular wave. Far from its origin a segment of a circle that crosses the boat trajectory can be approximated as a straight wave propagating in the same direction as the boat. Given that the wave pattern is "attached" to the boat, the crest velocity (phase velocity) where the waves cross the boat trajectory must equal the boat speed v_0. (If the crest traveled at any other speed the pattern would change with time.) Thus we expect the most important contributors to the transverse waves to be those having phase velocity in a narrow band centered on the boat speed v_0. That means the wavelengths will be in a narrow band centered on the wavelength λ_1 given, according to Eq. (4), by

$$\lambda_1 = 2\pi v_0^2/g. \tag{6}$$

Consider the boat as it moves along its trajectory. When it crosses a given point B_1 it sets that point oscillating. Since we assume wavelength λ_1 is the important one, that means the important frequency component of the oscillation of that point of water is at frequency f_1 given by

$$f_1 = \omega_1/2\pi = v_0/\lambda_1 = g/2\pi v_0. \tag{7}$$

(For a boat speed of 10 mph, $f_1 = 0.35$ Hz, i.e., one oscillation per 2.9 s.) The boat moves on, leaving the water point at B_1 oscillating at frequency f_1 (and neighboring frequencies) and emitting circular waves from B_1, with wavelength in a band centered on λ_1. Now consider a later time when the boat has moved on to a new position B_2 a distance from B_1 that we will call $2L$. The point at B_1 has been oscillating all this time. The wave crest with wavelength λ_1 emitted at the instant B_1 was set in oscillation is the "canonical circular wave." It has phase velocity v_0. Its arc passes through the present position of the boat, B_2. Between the canonical wave and the point B_1 there are many circular crests propagating from B_1. But because of destructive interference within the wavelength band the canonical crest and these other crests are all invisible, except for those in a region half way between B_1 and B_2.

Thus we have the rule that any curved wave you see following the boat and lagging behind the boat a distance L has as its most important parent (at that moment) a point B_1 located a distance $2L$ behind the boat. Therefore the wave has a radius of curvature R given by $R = L$. (In the more rigorous theory^5 the curved waves are more complicated than arcs of circles, but their radius of curvature at the boat trajectory agrees with our result $R = L$.)

If you look at this same crest a short time later, it is still at the same distance L behind the boat, and its present "most important parent" is a new point B_1 located a distance $2L$ behind the boat. The crest is continually being regenerated by new parent disturbances left behind by the boat.

VII. HOW WIDE ARE THE FEATHERLET WAVES?

That question is equivalent to the question, how many featherlet waves do you cross, near G, if you describe the line from B_1 to W in Fig. 3? We leave it as an exercise for the student. (I find the number of featherlets crossed should be about $1.5\,N^{1/2}$, where N is the wavelet number, starting at the boat and counting featherlets crossed as one progresses back along the wake line.)

VIII. COMPARISON WITH EXPERIMENT

The best photographs I have found so far are Fig. 8.2.4 of Ref. 5 and Fig. 51-10 of Ref. 1. The latter has large foreshortening and cannot be used for angle measurements. With the former I made the following crude measurements: $\theta_0 = 17°$ (theory says 19°); $\phi_0 = 51°$ (theory says 55°); wavelength ratio of transverse waves to V "featherlet" waves = 1.6 (theory says 1.5). To determine the radius of curvature R of a curved wave following the boat at distance L, I measured chord lengths C and sagittae S to find $R = C^2/8S + S/2$. Theory says $R = L$. I could only find a few decently measurable waves, and found $R/L = 0.8$ in one case and 1.1 in another. Thus there is fair agreement between theory and experiment.

As for the number of featherlet waves in a featherlet wavepacket, my crude theoretical estimate, $1.5N^{1/2}$, can be compared with Fig. 51-10 of Ref. 1, since the counting of wavelets is not affected by the foreshortening problem. Counting back $N = 9$ wavelets from the boat I count three or four featherlets along the featherlet propagation line, to be compared with a predicted 4.5. Counting back to $N = 20$ I count 6 or 7 featherlets, to be compared with a predicted 6.7. (The reader can easily repeat this measurement, since everyone has, or should have, a copy of Feynman.) The agreement is not bad. But it would be nice to have a very good photograph, where one could count up to perhaps $N = 100$.

ACKNOWLEDGMENTS

I thank Richard Muller and Charles Wohl for helpful comments and suggestions.

APPENDIX: FINDING THE MAXIMUM WAKE ANGLE

Choose units in Fig. 3 such that $B_2W = 1$, and $B_1G = GW = a$. Then by inspection we have

$$\theta = \tan^{-1} 2a,$$

and

$$\phi - \theta = \tan^{-1} a;$$

hence

$$\theta = \tan^{-1} 2a - \tan^{-1} a. \tag{A1}$$

To find the maximum value of θ, differentiate with respect to a, and set the result to zero:

$$\frac{d\theta}{da} = 2/(1 + 4a^2) - 1/(1 + a^2) = 0,$$

so that

$$a = (2)^{-1/2}.$$

Therefore

$$\theta = \tan^{-1}(2)^{1/2} - \tan^{-1}(2)^{-1/2} = 19.47°, \tag{A2}$$

and

$$\phi = \tan^{-1}(2)^{1/2} = 54.74°. \tag{A3}$$

Our results (A2) and (A3) agree with the more rigorous theory.[5]

Note that this problem is equivalent to the purely geometric one of finding that right triangle for which the median drawn from one acute-angled vertex to the center of the opposite side make the largest angle with the hypotenuse, with solution

adjacent side: opposite side: hypotenuse
$= 1:(2)^{1/2}:(3)^{1/2}$.

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[1] R. P. Feynman, R. B. Leighton, and M. Sands, *The Feynman Lectures on Physics* (Addison-Wesley, Reading, MA, 1965), Vol. 1, Chap. 51.

[2] Unfortunately the photograph in Fig. 51-10 of Ref. 1 is highly foreshortened, because it was not taken from directly overhead. The total opening angle of the V thus appears to be nearly 90° rather than the expected 38°. The foreshortening also exaggerates the "straightness" of the curved transverse waves between the arms of the V. A photograph with no apparent foreshortening, but with other shortcomings, appears in Fig. 8.2.4. of Ref. 5.

[3] Lord Kelvin (Sir W. Thomson), Proc. R. Soc. London **42**, 80 (1887).

[4] Sir Horace Lamb, *Hydrodynamics* (Dover, New York, 1945), Chap. IX, paragraph 256.

[5] J. J. Stocker, *Water Waves* (Interscience, New York, 1957), Chap. 8.

[6] I was stimulated to find this derivation by a challenge from Professor Richard M. Muller, who told me he heard there was an easy derivation in Feynman's text, but that I should be able to figure it out without looking in Feynman. After figuring it out I looked in Ref. 1 but found only that this was one of the topics "too complicated to analyze in detail here." Then I looked at Refs. 3, 4, and 5 and found them difficult.

[7] For a relatively easy derivation see, e.g., F. S. Crawford, *Waves*, Berkeley Physics Course, Vol. 3 (McGraw-Hill, New York, 1968), Sec. 7.3.

[8] The famous factor of 1/2 can be demonstrated with a canoe in a lake. Rock the canoe ten times to generate a wave packet. Now count the number of wave crests in the packet. You will count not ten but five. By the time the tenth crest was generated the first five crests have disappeared out the front end of the packet. New crests are born at the rear of the packet, travel through the packet at twice the packet (group) velocity, and disappear at the front. Or see, e.g., F. S. Crawford, Am. J. Phys. **41**,1203 (1973).

PROBLEM

Consider a long copper wire having a radius of 1 mm and carrying a time-varying current $I = t$ amps. What is the induced *electric* field at a distance of 10 cm from the wire? (Solution is on page 849.)