HOOKE'S LAW AND ELASTIC CONSTANTS

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https://www.phys.uconn.edu/~rozman/Courses/P3102_26S/

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Strain tensor ($i, j = 1, 2, 3$):

$$u_{ij} = \frac{1}{2}\left(\frac{\partial u_j}{\partial x_i} + \frac{\partial u_i}{\partial x_j}\right) = \frac{1}{2}\left(\partial_i u_j + \partial_j u_i\right) = \frac{1}{2}\left(\nabla_i u_j + \nabla_j u_i\right) \tag{1}$$

$$u_{ij} = \begin{bmatrix} \dfrac{\partial u_1}{\partial x_1} & \dfrac{1}{2}\left(\dfrac{\partial u_1}{\partial x_2} + \dfrac{\partial u_2}{\partial x_1}\right) & \dfrac{1}{2}\left(\dfrac{\partial u_1}{\partial x_3} + \dfrac{\partial u_3}{\partial x_1}\right) \\ \dfrac{1}{2}\left(\dfrac{\partial u_1}{\partial x_2} + \dfrac{\partial u_2}{\partial x_1}\right) & \dfrac{\partial u_2}{\partial x_2} & \dfrac{1}{2}\left(\dfrac{\partial u_2}{\partial x_3} + \dfrac{\partial u_3}{\partial x_2}\right) \\ \dfrac{1}{2}\left(\dfrac{\partial u_1}{\partial x_3} + \dfrac{\partial u_3}{\partial x_1}\right) & \dfrac{1}{2}\left(\dfrac{\partial u_2}{\partial x_3} + \dfrac{\partial u_3}{\partial x_2}\right) & \dfrac{\partial u_3}{\partial x_3} \end{bmatrix}. \tag{2}$$

Hooke's law in isotropic matter ($i, j = 1, 2, 3$):

$$\sigma_{ij} = 2\mu\, u_{ij} + \lambda\, \delta_{ij} \sum_{k=1}^{3} u_{kk} \tag{3}$$

$$\sigma_{ij} = \begin{bmatrix} (2\mu+\lambda)\, u_{11} + \lambda\,(u_{22} + u_{33}) & 2\mu\, u_{12} & 2\mu\, u_{13} \\ 2\mu\, u_{21} & (2\mu+\lambda)\, u_{22} + \lambda\,(u_{11} + u_{33}) & 2\mu\, u_{23} \\ 2\mu\, u_{31} & 2\mu\, u_{32} & (2\mu+\lambda)\, u_{33} + \lambda\,(u_{11} + u_{22}) \end{bmatrix}. \tag{4}$$

Inverting Hooke's law ($i, j = 1, 2, 3$):

$$\begin{aligned} u_{ij} &= \frac{1}{2\mu}\sigma_{ij} - \frac{\lambda}{2\mu(3\lambda+2\mu)}\delta_{ij}\sum_{k=1}^{3}\sigma_{kk} \\ &= \frac{1+\nu}{E}\sigma_{ij} - \frac{\nu}{E}\delta_{ij}\sum_{k=1}^{3}\sigma_{kk}, \end{aligned} \tag{5}$$

$$u_{ij} = \begin{pmatrix} \frac{1}{E}\left[\sigma_{11} - \nu\left(\sigma_{22}+\sigma_{33}\right)\right] & \frac{1+\nu}{E}\sigma_{12} & \frac{1+\nu}{E}\sigma_{13} \\ \frac{1+\nu}{E}\sigma_{21} & \frac{1}{E}\left[\sigma_{22} - \nu\left(\sigma_{11}+\sigma_{33}\right)\right] & \frac{1+\nu}{E}\sigma_{23} \\ \frac{1+\nu}{E}\sigma_{31} & \frac{1+\nu}{E}\sigma_{32} & \frac{1}{E}\left[\sigma_{33} - \nu\left(\sigma_{11}+\sigma_{22}\right)\right] \end{pmatrix}. \tag{6}$$

Only 2 independent elastic constants are required to completely describe the properties of an isotropic elastic medium, however there are many pairs of different constants from which to choose.

The constants referred to below are:

- λ – Lamé coefficient, has no name
- μ – Lamé coefficient, shear modulus
- ν – Poisson's ratio
- E – Young's modulus
- K – bulk modulus

$$\lambda = \frac{2\mu\nu}{1-2\nu} = \frac{\mu(E-2\mu)}{3\mu-E} = K - \frac{2}{3}\mu = \frac{E\nu}{(1+\nu)(1-2\nu)} = \frac{3K\nu}{1+\nu} = \frac{3K(3K-E)}{9K-E} \tag{7}$$

$$\mu = \frac{\lambda(1-2\nu)}{2\nu} = \frac{3}{2}(K-\lambda) = \frac{E}{2(1+\nu)} = \frac{3K(1-2\nu)}{2(1+\nu)} = \frac{3KE}{9K-E} \tag{8}$$

$$\nu = \frac{\lambda}{2(\lambda+\mu)} = \frac{\lambda}{(3K-\lambda)} = \frac{E}{2\mu} - 1 = \frac{3K-2\mu}{2(3K+\mu)} = \frac{3K-E}{6K} \tag{9}$$

$$E = \frac{\mu(3\lambda+2\mu)}{\lambda+\mu} = \frac{\lambda(1+\nu)(1-2\nu)}{\nu} = \frac{9K(K-\lambda)}{3K-\lambda} = 2\mu(1+\nu) = \frac{9K\mu}{3K+\mu} = 3K(1-2\nu) \tag{10}$$

$$K = \lambda + \frac{2}{3}\mu = \frac{\lambda(1+\nu)}{3\nu} = \frac{2\mu(1+\nu)}{3(1-2\nu)} = \frac{\mu E}{3(3\mu - E)} = \frac{E}{3(1-2\nu)} \tag{11}$$

The following combinations appear frequently:

$$\frac{\mu}{\lambda+\mu} = 1 - 2\nu, \qquad \frac{\lambda}{\lambda+2\mu} = \frac{\nu}{1-\nu}. \tag{12}$$