Name: ______________________________

Date: ______________________________

Collaborators: ____

(Collaborators submit their individually written assignments together)

Question:	1	2	3	Total
Points:	25	25	35	85
Score:				

Instructor/grader comments:

Lagrangian mechanics

1. The top of a wheel of mass M and radius R is connected to a massless spring (at its equilibrium length) with spring constant k, as shown in Fig. 1. All the mass of the wheel is at its center. The wheel rolls without slipping.

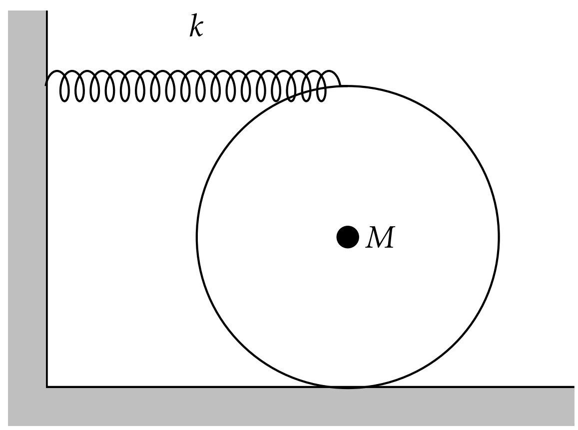
   (a) (5 points) How many degrees of freedom does the system have? List a suitable choice of generalized coordinate(s) that fully describes the configuration. (There is more than one option.)

   (b) (10 points) Write down the Lagrangian of the system in terms of your generalized coordinate(s). Assume that the linear amplitude of the rolling motion is much smaller than the radius of the wheel and the equilibrium length of the spring.

   Hint: note that as the wheel rolls, the displacement of its center and the displacement of its top are not equal.

   (c) (10 points) Derive the equation(s) of motion of the system. What is the frequency of small oscillations of the wheel?

Figure 1:



2. Two equal masses are glued to a massless hoop of radius R that is free to rotate about its center in a vertical plane. The angle between the masses is 2θ, as shown in Fig. 2. The acceleration of gravity is g.

   Let α be the angle of rotation of the hoop measured from its equilibrium (see Fig. 3). Use α as the generalized coordinate.

   (a) (15 points) What is the Lagrangian of the system?

   Hint: you may need to use the following trigonometric relation to simplify the Lagrangian:
$$\cos(x+y)+\cos(x-y)=2\cos(x)\cos(y).$$

   (b) (10 points) Derive the equation of motion of the system. What is the frequency of small oscillations of the hoop?

   To verify your expression for the frequency, consider the limit $\theta \to 0$, when you are supposed to recover the frequency of small oscillations of a simple pendulum, $\sqrt{g/R}$.

   Hints: Recall that for small α ($\alpha \ll 1$),
$$\cos\alpha \approx 1-\frac{\alpha^2}{2}, \qquad \sin\alpha \approx \alpha.$$

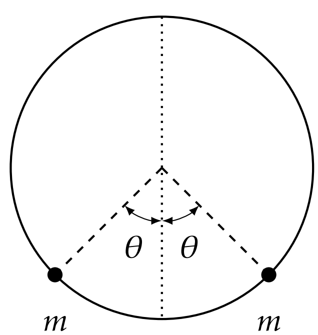


Figure 2: The equilibrium position of the hoop.

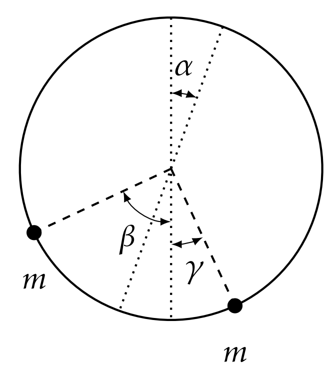


Figure 3: Rotated hoop; $\beta = \theta+\alpha$, $\gamma = \theta-\alpha$.

3. A bead is free to slide along a frictionless hoop of radius R. The hoop rotates with the constant angular speed ω around its vertical diameter (see Fig. 4).

   (a) (15 points) What is the Lagrangian of the system? Use the angle θ (see Fig. 4) as the generalized coordinate.

   Hint: Recall that the kinetic energy of the bead is

   $$T = \frac{1}{2} m R^2 \left(\dot{\theta}^2 + \omega^2 \sin^2(\theta) \right).$$

   (b) (5 points) Write down the equation of motion of the bead.

   (c) (15 points) What is the smallest angular speed such that the equilibrium position of the bead is not at the bottom of the hoop?

Figure 4: