P1

$$m = a^{\sigma} g^{\beta} \alpha^{\gamma}$$

(a) $$[m] = [a]^{\sigma} [g]^{\beta} [\alpha]^{\gamma}$$

$$[m] = M \;; \quad [a] = L \;; \quad [g] = L\cdot T^{-2}$$

$$[\alpha] = \left(\frac{N}{m}\right) = M\cdot T^{-2}$$

$$M = L^{\sigma}\cdot L^{\beta} T^{-2\beta}\cdot M^{\gamma}\cdot T^{-2\gamma}$$
$$= L^{\sigma+\beta}\, T^{-2(\beta+\gamma)}\cdot M^{\gamma}$$

$$\begin{cases} \sigma+\beta = 0 \rightarrow \sigma = 1 \\ \beta+\gamma = 0 \rightarrow \beta = -1 \\ \gamma = 1 \end{cases}$$

$$\boxed{m = \frac{a\cdot\alpha}{g}}$$

(b) $$\boxed{V = \frac{m}{\rho} = \frac{a\alpha}{\rho g}}$$

(c) $$V = \frac{2\cdot 10^{-3}\cdot 7\cdot 10^{-2}}{10^{3}\cdot 10} \quad \underbrace{\frac{m\cdot N/m}{kg/m^3\cdot m/s^2}}$$

$$\frac{\cancel{kg}\cdot m/\cancel{s^2}}{\cancel{kg}/(m^2\cdot\cancel{s^2})} = m^3$$

$$= \boxed{1.4\cdot 10^{-8}\, m^3}$$

Characteristic size of the drop:

$$\ell \sim \sqrt[3]{V} \approx \sqrt{14\cdot 10^{-9}\, m^3} \approx 2.4\cdot 10^{-2} m = 2.4\ mm$$

(d) $$V = a^{\sigma} g^{\beta} \alpha^{\gamma} \rho^{\delta}$$

$$L^{3} = L^{\sigma} \cdot L^{\beta} \cdot T^{-2\beta} M^{\gamma} T^{-2\gamma} M^{\delta} L^{-3\delta}$$

$$\begin{cases} \sigma + \beta - 3\delta = 3 \\ \beta + \gamma = 0 \\ \gamma + \delta = 0 \end{cases}$$

$\sigma + \beta - 3\delta = 3 \longrightarrow \sigma - 2\delta = 3$

$\beta + \gamma = 0 \longrightarrow \beta = \delta$

$\gamma + \delta = 0 \longrightarrow \gamma = -\delta$

3 equations,
4 unknowns

infinitely many solutions;
It is not possible to find the expression for V directly using dimensional analysis (as we learned it)

P2

(a) The dimension of potential (energy)

$[V(x)] = N \cdot m = M \cdot L \cdot T^{-2} \cdot L = M \cdot L^2 \cdot T^{-2}$

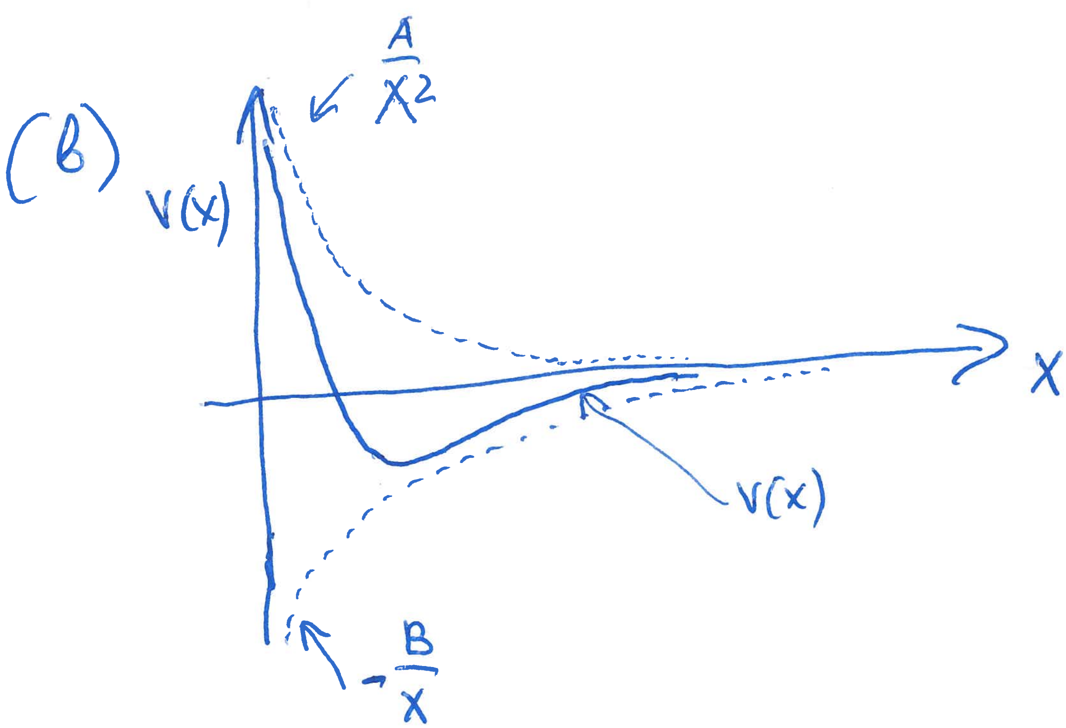
Thus:

$$\left[\frac{A}{x^2}\right] = \frac{[A]}{L^2} = M \cdot L^2 \cdot T^{-2}$$

$$[A] = M \cdot L^4 \cdot T^{-2}$$

$$\left[\frac{B}{x}\right] = \frac{[B]}{L} = M L^2 T^{-2} \rightarrow [B] = M \cdot L^3 \cdot T^{-2}$$

(b)



(c) $F(x) = -\dfrac{dV}{dx} = \boxed{+\dfrac{2A}{x^3} - \dfrac{B}{x^2}}$

$$[F(x)] = \frac{[A]}{L^3} + \frac{[B]}{L^2} = \frac{M \cdot L^4 \cdot T^{-2}}{L^3} + \frac{M \cdot L^3 \cdot T^{-2}}{L^2}$$

$$= \frac{M \cdot L}{T^2} = N$$

correct dimension of force

(d) Equilibrium: $F(x_0) = -\dfrac{dV}{dx}\Big|_{x=x_0} = 0$

$$\frac{2A}{x_0^3} = \frac{B}{x_0^2}; \quad \boxed{x_0 = \frac{2A}{B}}$$

$$[x_0] = \frac{[A]}{[B]} = \frac{\cancel{M} \cdot L^4 \cdot \cancel{T^{-2}}}{\cancel{M} \cdot L^3 \cancel{T^{-2}}} = L$$ - correct dimension

(e) In the vicinity of x_0

$$V(x_0 + s) = V(x_0) + \frac{1}{2}\frac{d^2V}{dx^2}\Big|_{x=x_0} \cdot s^2$$

$$\frac{d^2V}{dx^2} = \frac{d}{dx}\left(-\frac{2A}{x^3} + \frac{B}{x^2}\right) = \frac{6A}{x^4} - \frac{2B}{x^3}$$

from (c)

$$\left.\frac{d^2V}{dx^2}\right|_{x=x_0} = \frac{6A}{x_0^4} - \frac{2B}{x_0^3} =$$

$$= \frac{1}{x_0^3}\left(\frac{6A}{x_0} - 2B\right) = \frac{1}{x_0^3}\left(3B - 2B\right)$$

$$= \frac{B}{x_0^3} = \frac{B^4}{8A^3}$$

Potential energy in the vicinity of equilibrium:

$$V(x_0+s) \equiv V(s) = V_0 + \frac{1}{16}\frac{B^4}{A^3}\cdot s^2$$

irrelevant constant, can be dropped

Kinetic energy:

$$K = \frac{1}{2}m\dot{x}^2 = \frac{1}{2}m\dot{s}^2$$

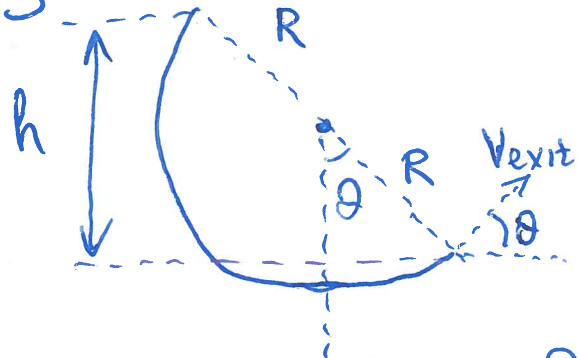
Total energy: $\underbrace{E = \frac{1}{2}m\dot{s}^2 + \frac{1}{16}\frac{B^4}{A^3}s^2}_{\text{energy of a harmonic oscillator.}}$

$$\omega^2 = \frac{\frac{1}{16}\frac{B^4}{A^3}}{\frac{1}{2}m} = \frac{1}{8}\frac{B^4}{A^3 m}; \quad \omega = \sqrt{\frac{B^4}{8A^3 m}}$$

$$[\omega] = \frac{[B]^2}{[A]^{3/2}[m]^{1/2}} = \frac{(M\cdot L^3\cdot T^{-2})^2}{(M\cdot L^4 T^{-2})^{3/2}\cdot M^{1/2}} = \frac{\cancel{M^2}\cancel{L^6}T^{-4}}{\cancel{M^{3/2}}\cdot\cancel{L^6}T^{-3}\cdot\cancel{M^{1/2}}}$$

$= T^{-1}$ – correct dimension for frequency

P3



$$\boxed{h = 2R\cos\theta}$$

(a)

Conservation of energy:

$mgh = \frac{1}{2} m V_{exit}^2$; $\quad V_{exit}^2 = 2gh = 4gR\cos\theta$

$$\boxed{V_{exit} = 2\sqrt{gR\cos\theta}}$$

The exit angle is $\boxed{\theta}$

(b) The range D:

$$D = \frac{V_{exit}^2 \cdot \sin(2\theta)}{g} = \frac{4\cancel{g}R\cos\theta \cdot 2\sin\theta\cos\theta}{\cancel{g}}$$

$$= 8\sin\theta \cdot \cos^2\theta \cdot R$$

correct dimension

(c) largest D $\longrightarrow \frac{dD}{d\theta} = 0$

$$\frac{d}{d\theta}\sin\theta \cdot \cos^2\theta = \cos\theta \cdot \cos^2\theta - \sin\theta \cdot 2\cos\theta \cdot \sin\theta$$

$$= \cos\theta\left(\cos^2\theta - 2\sin^2\theta\right)$$

$\cos\theta = 0$ corresponds to the smallest distance, $D=0$

$\cos^2\theta - 2\sin^2\theta = 0 \rightarrow$ $\boxed{\tan\theta = \frac{1}{\sqrt{2}} \quad \theta \approx 35°}$

$\downarrow$

(*) $\cos^2\theta = 2\sin^2\theta$

(**) $1 - \sin^2\theta - 2\sin^2\theta = 0$

$\sin\theta = \frac{1}{\sqrt{3}}$

Using (*) and (**)

$$D = 8 \cdot \sin\theta \cdot \underbrace{\cos^2\theta}_{2\sin^2\theta} \cdot R = 16R \underbrace{\sin^3\theta}_{3^{-3/2} = \frac{1}{3\sqrt{3}}}$$

$$\boxed{D = \frac{16}{3\sqrt{3}} R} \simeq 3.1\,R \approx 3R$$