Question:	1	2	3	Total
Points:	25	50	25	100

Show all your work and indicate your reasoning in order to receive the credit. Present your answers in *low-entropy* form.

Dimensional analysis

1. **Pressure in the center of a self-gravitating mass**

   (a) (5 points) What is the dimension of gravitational constant G? Use the Newton's law of gravity and show your derivation.

   $$F = G\frac{m_1 m_2}{r^2}.$$

   (b) (15 points) Use the dimensional analysis to find the pressure in the center of a large self-gravitating sphere of radius R of density ρ. The pressure (which has units of force per area) depends upon R, ρ, and gravitational constant G.

   (c) (5 points) How the pressure in the center is going to change if you double the radius of the sphere?

Solving differential equations of motion

2. At the time $t = 0$ a heavy sphere of mass m is released with zero initial velocity. The sphere is subject to the force of gravity, mg, and a drag force

   $$F(v) = -\beta v^2.$$

   (a) (10 points) Write the equation of motion of the sphere and provide the initial conditions.

   Chose the initial position of the sphere as the origin of your coordinate system. Direct the x axis along the direction of the fall.

   (b) (5 points) What is the dimension of the parameter β? Show your derivation.

   (c) (5 points) What is the maximal velocity of of the falling sphere?

   (d) (10 points) Find the speed of the particle vs time, $v(t)$.

   Hints: you may need to use the integral

   $$\int \frac{\mathrm{d}u}{1-u^2} = \operatorname{arctanh}(u).$$

For the reference, $\operatorname{arctanh}(0) = 0$.

(e) (10 points) Find the position of the particle vs time, $x(t)$.

Hints: you may need to use the integral

$$\int \tanh(u)\, \mathrm{d}u = \ln\left[\cosh(u)\right].$$

For the reference, $\cosh(0) = 1$.

(f) (10 points) Verify that your result for $x(t)$ make sense by considering the limits (a) $\beta \to 0$, m is finite; and (b) $m \to \infty$, β is finite.

Free body diagram

3. (25 points) In the Atwood's machine in Fig. 1, what should M be, in terms of m_1 and m_2, so that it doesn't move?

Hints: what is the tension in the rope in Pulley 1 when mass M is at rest? What is then the tension in the rope in Pulley 2?

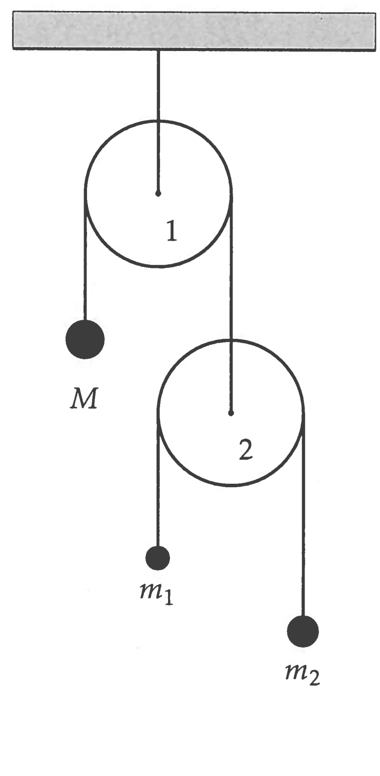

Figure 1: Atwood's machine

P1.

(a) $$G = \frac{F \cdot r^2}{m_1 m_2}$$

$$[G] = \frac{[F]\cdot[r^2]}{[m_1 m_2]} = \frac{\cancel{M} \cdot L \cdot T^{-2} \cdot L^2}{M^{\cancel{2}}} =$$

$$= L^3 \cdot T^{-2} \cdot M^{-1}$$

(b) $$P = R^{\alpha} \rho^{\beta} G^{\gamma}$$

$$[P] = [R^{\alpha}] \cdot [\rho^{\beta}][G^{\gamma}];$$

$$M \cdot \cancel{L} \cdot T^{-2} / L^{\cancel{2}} = L^{\alpha} \cdot M^{\beta} \cdot L^{-3\beta} \cdot L^{3\gamma} \cdot T^{-2\gamma} \cdot M^{-\gamma}$$

$$M \cdot T^{-2} L^{-1} = L^{\alpha+3\gamma-3\beta} \cdot M^{\beta-\gamma} \cdot T^{-2\gamma}$$

$$\begin{cases} \beta - \gamma = 1 & \longrightarrow \beta = \gamma + 1 = 2 \\ 2\gamma = 2 & \longrightarrow \gamma = 1 \\ \alpha + 3\gamma - 3\beta = -1 & \longrightarrow \alpha = 3\beta - 3\gamma - 1 = 2 \end{cases}$$

$$P = G \rho^2 R^2$$

(c) $R \to 2R \qquad P \to 4P$

P2.

(a) $$ma = mg - \beta v^2$$

$$\boxed{v(0) = 0}\ ; \quad \boxed{x(0) = 0}$$

$$a = \frac{dv}{dt}\ ; \qquad \boxed{\frac{dv}{dt} = g\left(1 - \frac{\beta}{mg} v^2\right)} \quad (*)$$

(b) $$[\beta] = \frac{[F]}{[v^2]} = \frac{M \cdot L \cdot T^{-2}}{(L \cdot T^{-1})^2} = \boxed{M \cdot L^{-1}}$$

(c) Maximal (terminal) velocity:

$$a = 0 \rightarrow mg = \beta v_t^2$$

$$\boxed{v_t = \left(\frac{mg}{\beta}\right)^{1/2}}$$

(d) Let's separate variables in (*):

$$\frac{dv}{1 - \frac{\beta}{mg} v^2} = g\,dt$$

$$\int_0^v \frac{dv'}{1 - \frac{\beta}{mg} v'^2} = g \int_0^t dt' = gt$$

$$\int_0^v \frac{dv'}{1-\frac{\beta}{mg}v'^2} =$$

$$\boxed{\begin{aligned} &\frac{\beta}{mg}v^2 = u^2; \quad u = \left(\frac{\beta}{mg}\right)^{1/2} v \\ &v = \left(\frac{mg}{\beta}\right)^{1/2} u \\ &dv = \left(\frac{mg}{\beta}\right)^{1/2} du \end{aligned}}$$

$$\left(\frac{mg}{\beta}\right)^{1/2}\int_0^u \frac{du}{1-u^2} =$$

$$= \left(\frac{mg}{\beta}\right)^{1/2} \operatorname{arctanh}(u) = \left(\frac{mg}{\beta}\right)^{1/2} \operatorname{arctanh}\left[\left(\frac{\beta}{mg}\right)^{1/2} v\right]$$

$$\left(\frac{mg}{\beta}\right)^{1/2} \operatorname{arctanh}\left[\left(\frac{\beta}{mg}\right)^{1/2} v\right] = gt$$

$$\operatorname{arctanh}\left[\left(\frac{\beta}{mg}\right)^{1/2} v\right] = \left(\frac{\beta g}{m}\right)^{1/2} t$$

$$\left(\frac{\beta}{mg}\right)^{1/2} v = \tanh\left[\left(\frac{\beta g}{m}\right)^{1/2} t\right]$$

$$\boxed{v(t) = \left(\frac{mg}{\beta}\right)^{1/2} \cdot \tanh\left[\left(\frac{\beta g}{m}\right)^{1/2} t\right]}$$

(e) $$x(t) = \int_0^t v(t')\,dt'$$

$$= \left(\frac{mg}{\beta}\right)^{1/2} \int_0^t \tanh\left[\left(\frac{\beta g}{m}\right)^{1/2} t\right] dt =$$

$$w = \left(\frac{\beta g}{m}\right)^{1/2} t \; ; \quad dt = \left(\frac{m}{\beta g}\right)^{1/2} dw$$

$$= \left(\frac{mg}{\beta}\right)^{1/2} \cdot \left(\frac{m}{\beta g}\right)^{1/2} \int_0^w \tanh(w)\,dw =$$

$$= \boxed{\frac{m}{\beta} \ln\left[\cosh\left[\left(\frac{\beta g}{m}\right)^{1/2} t\right]\right] = x(t)}$$

(f) For $\varepsilon \ll 1$: $\cosh(\varepsilon) \approx 1 + \frac{\varepsilon^2}{2}$

$$\ln\left(1 + \frac{\varepsilon^2}{2}\right) \approx \frac{\varepsilon^2}{2}$$

When $\beta \to 0$, or $m \to \infty$

$\left(\frac{\beta g}{m}\right)^{1/2} \to 0$

$$x(t) = \frac{m}{\beta} \cdot \left[\left(\frac{\beta g}{m}\right)^{1/2} t\right]^2 \cdot \frac{1}{2} = \frac{g t^2}{2}$$

$\swarrow$ simple motion with constant acceleration g

P3

1. M is at rest $\longrightarrow$ $T_1 = Mg$

2. Puley 2 is at rest $\rightarrow$ $T_1 = 2T_2$

$$T_2 = \frac{Mg}{2}$$

3. Newton's equations for masses 1 and 2

$$\begin{cases} m_1 a = m_1 g - T_2 = m_1 g - \frac{Mg}{2} \\ -m_2 a = m_2 g - T_2 = m_2 g - \frac{Mg}{2} \end{cases}$$

$$+\begin{cases} a = g - \frac{M}{m_1}\frac{g}{2} \\ -a = g - \frac{M}{m_2}\frac{g}{2} \end{cases}$$

$$0 = 2\cancel{g} - \frac{M\cancel{g}}{2}\left(\frac{1}{m_1} + \frac{1}{m_2}\right)$$

$$M = \frac{4}{\frac{1}{m_1} + \frac{1}{m_2}} = \boxed{\frac{4 m_1 m_2}{m_1 + m_2}}$$