P1

(a) $[P] = [J/s] = M \cdot L^2 \cdot T^{-3}$

$$P = r^{\alpha} v^{\beta} \rho^{\gamma}$$

$$M L^{+2} T^{-3} = L^{\alpha} \cdot L^{\beta} \cdot T^{-\beta} M^{\gamma} \cdot L^{-3\gamma}$$

$$\begin{cases} 1 = \gamma & \longrightarrow \gamma = 1 \\ -3 = -\beta & \longrightarrow \beta = 3 \\ +2 = \alpha + \beta - 3\gamma & \longrightarrow \alpha = 2 \end{cases}$$

$$\boxed{P = r^2 v^3 \rho}$$

$v \to 2v$, $P \to 8P$

P2 (a)

Θ – dimensionless

$$\Theta = R^{\alpha} M G^{\beta} c^{\gamma}$$

$[G] = M^{-1} L^3 T^{-2}$ $\quad$ $[c] = L \cdot T^{-1}$

$$1 = L^{\alpha} \cdot M^{1} \cdot M^{-\beta} L^{3\beta} \cdot T^{-2\beta} \cdot L^{\gamma} \cdot T^{-\gamma}$$

$$\left\{ \begin{array}{ll} 1 - \beta = 0 & \longrightarrow \beta = 1 \\ -2\beta - \gamma = 0 & \longrightarrow \gamma = -2 \\ \alpha + 3\beta + \gamma = 0 & \quad\;\; \alpha = -1 \end{array} \right\} \to \boxed{\Theta = \frac{MG}{Rc^2}}$$

P3 The speed of a rocket starting from rest

$$v = u \ln\left(\frac{M}{m}\right),$$

where m is the 'current' mass

Linear momentum of the rocket:

$$p = mv = mu \ln\left(\frac{M}{m}\right)$$

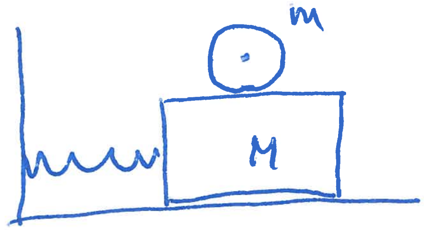
It is at maximum when $\frac{dp}{dm} = 0$

$$\frac{dp}{dm} = \cancel{u} \ln\left(\frac{M}{m}\right) + \cancel{m}\cancel{u}\frac{1}{\cancel{M}/\cancel{m}}\left(-\frac{\cancel{M}}{\cancel{m^2}}\right) = 0;$$

$$\ln\left(\frac{M}{m}\right) = 1; \quad \frac{M}{m} = e; \quad \boxed{m = \frac{M}{e}}$$

$$\boxed{p_{max} = \frac{Mu}{e}}$$

P4



(a) $T_1 = \frac{1}{2} M \dot{x}^2$

$V_1 = \frac{1}{2} k x^2$

(b) No-slip condition: the point of contact between the block and the cylinder moves with the linear velocity of the cylinder:

$$V_c - r\dot{\varphi} = \dot{x}$$

(c) $T_t = \frac{1}{2} m V_c^2 = \frac{1}{2} m (\dot{x} + r\dot{\varphi})^2$

$T_r = \frac{1}{2} I \dot{\varphi}^2 = \frac{1}{4} m r^2 \dot{\varphi}^2$

$T_2 = \frac{1}{2} m \dot{x}^2 + \frac{3}{4} m r^2 \dot{\varphi}^2 + r m \dot{x} \dot{\varphi}$

(d) $L = T_1 + T_2 - V_1 =$

$$= \frac{1}{2}(M+m)\dot{x}^2 + \frac{3}{4} m r^2 \dot{\varphi}^2 + r m \dot{x} \dot{\varphi} - \frac{1}{2} k x^2$$

(e) $$\frac{d}{dt}\frac{\partial L}{\partial \dot{\varphi}} = \frac{3}{2} m r^2 \ddot{\varphi} + r m \ddot{x}$$

$$\frac{\partial L}{\partial \varphi} = 0$$

$$\frac{3}{2} \cancel{m} r \cancel{r} \ddot{\varphi} + \cancel{r} \cancel{m} \ddot{x} = 0$$

$$\frac{3}{2} r \ddot{\varphi} + \ddot{x} = 0 \qquad (*)$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{x}} = (M+m)\ddot{x} + r m \ddot{\varphi}$$

$$\frac{\partial L}{\partial x} = -kx$$

$$(M+m)\ddot{x} + r m \ddot{\varphi} + kx = 0 \qquad (**)$$

(f) Use $(*)$: $$\ddot{\varphi} = -\frac{2}{3r}\ddot{x}$$

Plug this into $(**)$:

$$(M+m)\ddot{x} - \frac{2}{3} m \ddot{x} + kx = 0$$

$$\boxed{\left(M + \frac{1}{3} m\right)\ddot{x} + kx = 0}$$

harmonic oscillator

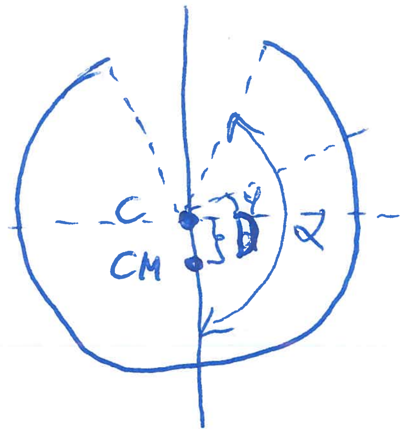
(g) $$\omega^2 = \frac{k}{M + \frac{1}{3}m}$$ 5

In the limit $m \to 0$, $\omega = \sqrt{\frac{k}{M}}$

correct!

P5 $\quad$ 6

(a)



linear density $\rho = \dfrac{M}{2\alpha R}$

Due to symmetry of the system the center of mass is on the 'vertical' axis. We need to calculate D only.

$$dm = \rho R \, d\varphi$$

$$D = \frac{2}{M}\int_{-\pi/2}^{\alpha-\frac{\pi}{2}} R\sin\varphi \, dm = \frac{\cancel{2}}{\cancel{M}} \cdot \frac{\cancel{M} R^{\cancel{2}}}{\cancel{2}\alpha \cancel{R}} \int_{-\frac{\pi}{2}}^{\alpha-\frac{\pi}{2}} \sin\varphi \, d\varphi =$$

$$= -\frac{R}{\alpha}\cos\varphi \Big|_{-\frac{\pi}{2}}^{\alpha-\frac{\pi}{2}} = -\frac{R}{\alpha}\cos\left(\alpha-\frac{\pi}{2}\right) = -\frac{R}{\alpha}\sin\alpha$$

In the limit $\alpha \to \pi$, $D \to 0$, i.e. the center of mass is in the center of the circle, as it should be. In the limit, $\alpha \to 0$, $D \to -R$, as it should be, since the mass is concentrated at the 'bottom' of the circle.

From now on $\quad D \to |D|$, $\quad \boxed{D = \frac{R}{\alpha}\sin\alpha}$

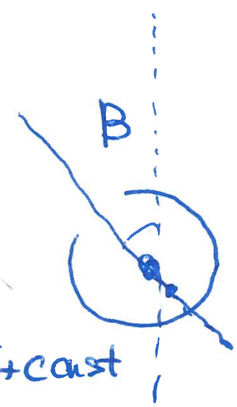
(b) The moment of inertia of the arc about the point C (its 'geometric' center) is

$$I_c = MR^2$$

Parallel axis theorem:

$$I_c = I_{cm} + MD^2$$

$$\boxed{\begin{aligned} I_{cm} &= MR^2 - MD^2 \\ &= MR^2\left(1 - \frac{\sin^2\alpha}{\alpha^2}\right) \end{aligned}}$$



(c) Potential energy

$$V(\beta) = -MgD\cos\beta \approx \tfrac{1}{2}MgD\beta^2 + \text{const}$$

Kinetic energy, in the limit of small oscillations,

$$T = \frac{1}{2}M\dot{\beta}^2(R-D)^2 \quad !$$

Discussed in class

$$L = \frac{1}{2}M\dot{\beta}^2(R-D)^2 - \frac{1}{2}MgD\beta^2$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\beta}} = M(R-D)^2\ddot{\beta} \; ; \quad \frac{\partial L}{\partial \beta} = -MgD\beta$$

Equation of motion:

$$\cancel{M}(R-D)^2\ddot{\beta} + \cancel{M}gD\beta = 0$$ — harmonic oscillator

$$\ddot{\beta} + \frac{gD}{(R-D)^2}\beta = 0$$

$$\omega^2 = \underbrace{\left(\frac{g}{R-D}\right)}_{\text{dimension of frequency squared}} \cdot \underbrace{\left(\frac{D}{R-D}\right)}_{\text{dimensionless ratio}}$$

In the limit $\alpha \to \pi$, $D \to 0$, $\omega \to 0$

Indeed, the 'whole' circle is not going to oscillate.