INTEGRATION OF FAST OSCILLATING FUNCTIONS

Spring semester 2025

https://www.phys.uconn.edu/~rozman/Courses/P2400_25S/

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1 The method of stationary phase

There is an immediate generalization of the Laplace integrals

$$\int_a^b f(t)\, e^{x\phi(t)}\, \mathrm{d}t \tag{1}$$

which we obtain by allowing the function $\phi(t)$ in Eq. (1) to be complex. We may assume that $f(t)$ is real; if it were complex, $f(t)$ could be decomposed into a sum of its real and imaginary parts. However, allowing $\phi(t)$ to be complex poses nontrivial problems. We consider the special case in which $\phi(t)$ is pure imaginary: $\phi(t) = i\psi(t)$ where $\psi(t)$ is real. The resulting integral

$$I(x) = \int_a^b f(t)\, e^{ix\psi(t)}\, \mathrm{d}t \tag{2}$$

with $f(t)$, $\psi(t)$, a, b, x all real is called a generalized Fourier integral. When $\psi(t) = t$, $I(x)$ is an ordinary Fourier integral.

The method of stationary phase gives the leading asymptotic behavior of generalized Fourier integrals having stationary points, $\psi' = 0$. This method is similar to Laplace's method in that the leading contribution to $I(x)$ comes from a small interval surrounding the stationary points of ψ.

1.1 Contributions from stationary points

To evaluate the integral

$$F = \int_0^\infty u^q \, e^{i\lambda u^p} \, \mathrm{d}u, \tag{3}$$

where λ, p, and q are real, $\lambda > 0$, $p > 1$, $q \geq 0$, consider the following contour integral,

$$J = \oint_C z^q \, e^{i\lambda z^p} \, \mathrm{d}z, \tag{4}$$

where the integration contour C is sketched in Fig. 1. The integrand in Eq. (4) is analytic inside the contour C, thus $J = 0$. On the other hand,

$$J = J_I + J_{II} + J_{III}, \tag{5}$$

where J_I is the integral along the positive real axis, J_{II} is the integral along the circular arc of the radius $R \to \infty$, an J_{III} is the integral (from infinity to the origin) along the ray making the angle $\frac{\pi}{2p}$ with the real axis. Notice first that

$$F = J_I. \tag{6}$$

Next,

$$J_{II} = 0. \tag{7}$$

The proof of Eq. (7) is similar to the proof of Jordan's lemma.

Finally, on the integration path for J_{III}:

$$z = re^{i\frac{\pi}{2p}}, \quad \mathrm{d}z = e^{i\frac{\pi}{2p}}\mathrm{d}r, \quad z^p = r^p e^{i\frac{\pi}{2}} = ir^p, \quad z^q = r^q e^{iq\frac{\pi}{2p}}. \tag{8}$$

Therefore

$$J_{III} = e^{i(q+1)\frac{\pi}{2p}} \int_\infty^0 r^q \, e^{-\lambda r^p} \, \mathrm{d}r = -e^{i(q+1)\frac{\pi}{2p}} \int_0^\infty r^q \, e^{-\lambda r^p} \, \mathrm{d}r. \tag{9}$$

The last integral can be evaluated by introducing the new integration variable

$$u = \lambda r^p, \quad r = \lambda^{-\frac{1}{p}} u^{\frac{1}{p}}, \quad \mathrm{d}r = \frac{1}{p}\lambda^{-\frac{1}{p}} u^{\frac{1}{p}-1}\mathrm{d}u, \quad r = \lambda^{-\frac{q}{p}} u^{\frac{q}{p}}. \tag{10}$$

Thus,

$$\int_0^\infty r^q \, e^{-\lambda r^p} \, \mathrm{d}r = \frac{1}{p}\lambda^{-\frac{q+1}{p}} \int_0^\infty e^{-u} u^{\frac{q+1}{p}-1} \mathrm{d}u = \frac{1}{p}\lambda^{-\frac{q+1}{p}} \Gamma\left(\frac{q+1}{p}\right). \tag{11}$$

Combining Eqs. (5), (7), (9), (11), we obtain:

$$\int_0^\infty u^q\, e^{i\lambda u^p}\, \mathrm{d}u = \frac{1}{p} e^{i(q+1)\frac{\pi}{2p}} \lambda^{-\frac{q+1}{p}}\, \Gamma\left(\frac{q+1}{p}\right). \tag{12}$$

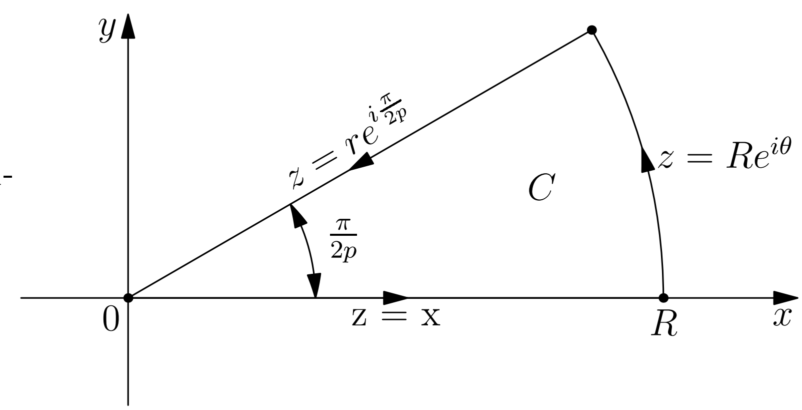
Similarly,

$$\int_0^\infty u^q\, e^{-i\lambda u^p}\, \mathrm{d}u = \frac{1}{p} e^{-i(q+1)\frac{\pi}{2p}} \lambda^{-\frac{q+1}{p}}\, \Gamma\left(\frac{q+1}{p}\right). \tag{13}$$

For the important particular case $p = 2$, $q = 0$:

$$\int_0^\infty e^{\pm i\lambda u^2}\, \mathrm{d}u = e^{\pm i\frac{\pi}{4}} \lambda^{-\frac{1}{2}}\, \frac{\sqrt{\pi}}{2}. \tag{14}$$

Figure 1: The integration contour for Eq. (4).



1.2 Examples

Example 1. Find the leading term of the asymptotics of the following integral for $\lambda \to \infty$:

$$I(\lambda) = \int_{-4}^{5} \cos\left(\lambda \sinh^2(x)\right) \sqrt{1 + x^2}\, e^{-\frac{t^2}{6}}\, \mathrm{d}x. \tag{15}$$

Since only small $|x|$, such that $|x| \sim \frac{1}{\sqrt{\lambda}} \ll 1$ are important,

$$\sinh x \sim x, \tag{16}$$

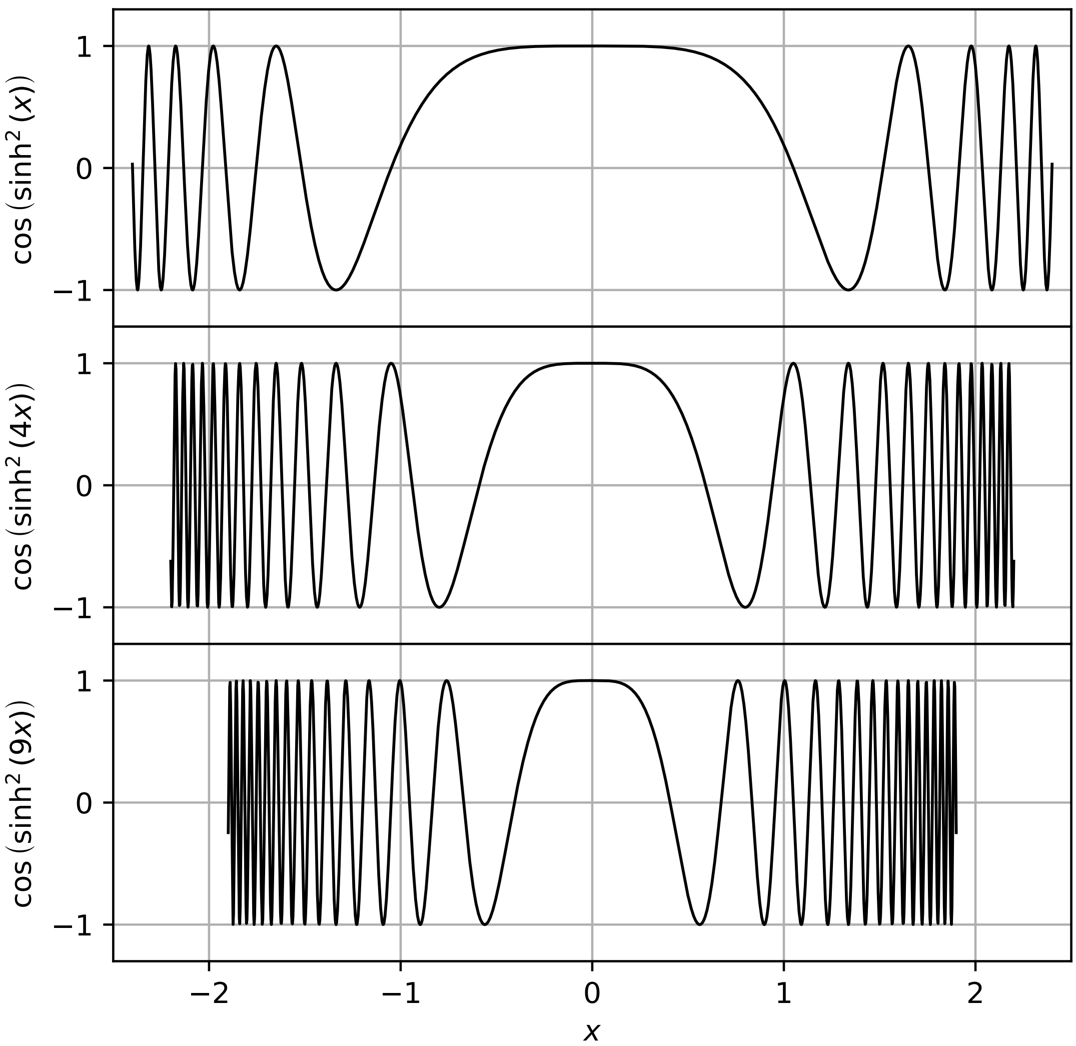


Figure 2: The graphs of the oscillating factor, $\cos\left(\lambda \sinh^2(x)\right)$ in Eq. (15), for $\lambda = 1, 4, 9$.

$$\cos\left(\lambda \sinh^2(x)\right) \sim \cos\left(\lambda x^2\right) \tag{17}$$

$$\sqrt{1+x^2}\, e^{-\frac{t^2}{6}} \sim 1. \tag{18}$$

$$I(\lambda) \approx \mathrm{Re} \int_{-4}^{5} e^{i\lambda x^2}\, \mathrm{d}x \approx \mathrm{Re} \int_{-\infty}^{\infty} e^{i\lambda x^2}\, \mathrm{d}x. \tag{19}$$

New integration variable,

$$u^2 = \lambda x^2 \quad \longrightarrow \quad x^2 = \frac{u^2}{\lambda} \quad \longrightarrow \quad x = \frac{u}{\sqrt{\lambda}} \quad \longrightarrow \quad \mathrm{d}x = \frac{1}{\sqrt{\lambda}}\, \mathrm{d}u. \tag{20}$$

$$I(\lambda) = \mathrm{Re} \frac{1}{\sqrt{\lambda}} \underbrace{\int_{-\infty}^{\infty} e^{iu^2}\, \mathrm{d}u}_{\sqrt{\pi} e^{i\frac{\pi}{4}}} = \sqrt{\frac{\pi}{\lambda}}\, \underbrace{\mathrm{Re}\left(e^{i\frac{\pi}{4}}\right)}_{\frac{1}{\sqrt{2}}} = \boxed{\sqrt{\frac{\pi}{2\lambda}}} \tag{21}$$

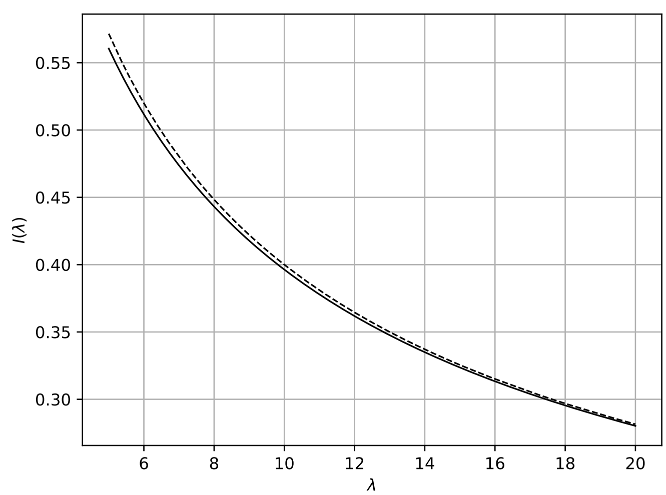


Figure 3: Asymptotics Eq. (21) (solid line) compared to numerically evaluated Eq. (15) (dashed line) for $5 \le \lambda \le 20$.

Example 2. Find the leading term of the asymptotics of the Bessel function $J_0(x)$ for $x \to \infty$:

$$J_0(x) \equiv \frac{1}{\pi} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos(x \cos\theta)\, \mathrm{d}\theta. \tag{22}$$

Recall that the Bessel function $J_0(x)$ is a solution of the following second order linear differential equation:

$$xy'' + y' + xy = 0. \tag{23}$$

Equation (23) belongs to the type that can be solved using Laplace method for ordinary differential equations. Using the same notations that we used in the relevant handout, we have here:

$$a_2 = 0,\ b_2 = 1,\ a_1 = 1,\ b_1 = 0,\ a_0 = 0,\ b_0 = 1. \tag{24}$$

$$P(t) \equiv \sum_n a_n t^n = t, \qquad Q(t) \equiv \sum_n b_n t^n = 1 + t^2, \tag{25}$$

$$\int \frac{P(t)}{Q(t)} \mathrm{d}t = \int \frac{t\, \mathrm{d}t}{1+t^2} = \frac{1}{2} \int \frac{\mathrm{d}(1+t^2)}{1+t^2} = \ln\left(1+t^2\right)^{\frac{1}{2}}. \tag{26}$$

Thus

$$Z \equiv \frac{1}{Q(t)} \exp\left(\int \frac{P(t)}{Q(t)} \mathrm{d}t \right) = \left(1+t^2\right)^{-\frac{1}{2}}. \tag{27}$$

The contour integral over yet unspecified contour C,

$$y(x) = \int_C e^{xt} \left(1+t^2\right)^{-\frac{1}{2}} \mathrm{d}t, \tag{28}$$

is therefore a solution of Eq. (23) if the function

$$e^{xt} Q(t) Z(t) = e^{xt} \left(1+t^2\right)^{\frac{1}{2}} \tag{29}$$

takes on the same values at the ends of the integration contour C.

Lets chose contour C as the one connecting the points $t_i = -i$ and $t_f = i$. The values of the function Eq. (29) are equal (and equals to 0) as required.

Therefore,

$$y(x) = \int_{-i}^{i} e^{xt} \left(1+t^2\right)^{-\frac{1}{2}} \mathrm{d}t \tag{30}$$

is the solution of the Bessel equation Eq. (23). Let's change the integration variable as following:

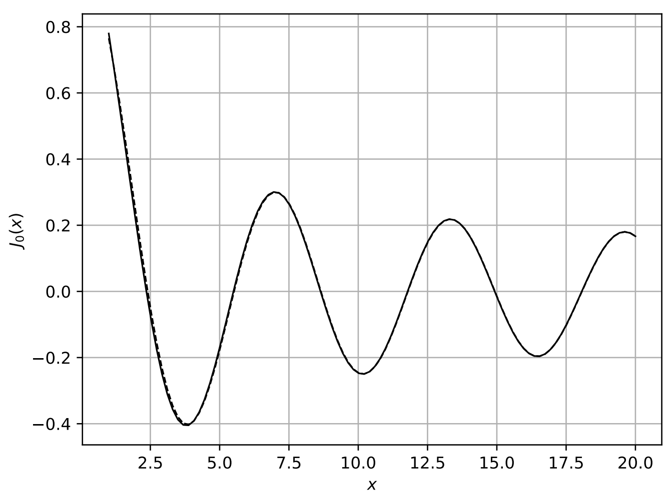
$$t = -i\cos\theta, \quad 0 \le \theta \le \pi, \quad \mathrm{d}t = i\sin\theta\,\mathrm{d}\theta \quad \left(1+t^2\right)^{-\frac{1}{2}} = \frac{1}{\sin\theta}. \tag{31}$$

Thus,

$$y(x) \sim \int_0^{\pi} e^{-ix\cos\theta}\,\mathrm{d}\theta = \int_0^{\pi} \cos(x\cos\theta)\,\mathrm{d}\theta = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos\left(x\cos\theta\right)\mathrm{d}\theta, \tag{32}$$

which is (up to a multiplicative constant) integral Eq. (22).

Figure 4: Asymptotics Eq. (35) (solid line) compared to numerically evaluated Eq. (22) (dashed line) for $1 \le x \le 20$.



Returning to the original problem, let's rewrite integral Eq. (22) in the exponential form:

$$J_0(x) = \frac{1}{\pi}\,\mathrm{Re}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} e^{ix\cos\theta}\mathrm{d}\theta. \tag{33}$$

The stationary point of the phase factor is at $\theta = 0$. Only small θ contribute to the integral. Therefore.

$$\cos\theta \approx 1 - \frac{\theta^2}{2}. \tag{34}$$

$$
\begin{aligned}
J_0(x) &\sim \frac{1}{\pi}\mathrm{Re}\left[e^{ix}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} e^{-i\frac{x\theta^2}{2}}\mathrm{d}\theta\right] \sim \frac{1}{\pi}\sqrt{\frac{2}{x}}\mathrm{Re}\left[e^{ix}\int_{-\infty}^{\infty} e^{-i\frac{x}{2}\theta^2}\mathrm{d}\left(\sqrt{\frac{x}{2}}\theta\right)\right] \\
&= \frac{1}{\pi}\sqrt{\frac{2}{x}}\mathrm{Re}\left[e^{ix}\sqrt{\pi}e^{-i\frac{\pi}{4}}\right] = \boxed{\sqrt{\frac{2}{\pi x}}\cos\left(x-\frac{\pi}{4}\right)}
\end{aligned} \tag{35}
$$

Example 3. Find the leading term of the asymptotics of the Airy function, $\mathrm{Ai}(x)$, for $x \to -\infty$:

$$
\mathrm{Ai}(x) \equiv \frac{1}{\pi}\int_0^{\infty}\cos\left(kx+\frac{k^3}{3}\right)\mathrm{d}k \tag{36}
$$

Recall that the Airy function $\mathrm{Ai}(x)$ is a solution of the following second order linear differential equation:

$$
y'' - xy = 0. \tag{37}
$$

For $x < 0$ lets rewrite Eq. (36) as follows:

$$
\mathrm{Ai}(x) = \frac{1}{\pi}\mathrm{Re}\int_0^{\infty} e^{i\left(-k|x|+\frac{k^3}{3}\right)}\mathrm{d}k = \frac{1}{\pi}\mathrm{Re}\int_0^{\infty} e^{i\phi(k)}\mathrm{d}k, \tag{38}
$$

where we introduced the notation

$$
\phi(k) = -k|x| + \frac{k^3}{3}. \tag{39}
$$

The position of the stationary point of the phase factor is determined from the relation

$$
\frac{\mathrm{d}\phi}{\mathrm{d}k} = -|x| + k^2 = 0, \tag{40}
$$

i.e.

$$
k_0 = \sqrt{|x|}. \tag{41}
$$

We have a case a moving stationary point. Let's introduce a new integration variable, u:

$$
u = |x|^{-\frac{1}{2}}k, \quad k = |x|^{\frac{1}{2}}u, \quad \mathrm{d}k = |x|^{\frac{1}{2}}\mathrm{d}u, \quad k|x| = |x|^{\frac{3}{2}}u, \quad k^3 = |x|^{\frac{3}{2}}u^3. \tag{42}
$$

$$
\phi(u) = |x|^{\frac{3}{2}}\left(-u+\frac{u^3}{3}\right). \tag{43}
$$

The position of the stationary point is now a constant:

$$\frac{\mathrm{d}\phi}{\mathrm{d}u} = |x|^{\frac{3}{2}}\left(-1+u^2\right) = 0 \quad \rightarrow \quad u_0 = 1. \tag{44}$$

The Taylor expansion of $\phi(u)$ in the vicinity of u_0 is:

$$\phi(u) = -\frac{2}{3}|x|^{\frac{3}{2}} + |x|^{\frac{3}{2}}(u-1)^2. \tag{45}$$

$$\mathrm{Ai}(x) = \frac{1}{\pi}\mathrm{Re}\left[e^{-i\frac{2}{3}|x|^{\frac{3}{2}}}|x|^{\frac{1}{2}}\int_0^\infty e^{i|x|^{\frac{3}{2}}(u-1)^2}\mathrm{d}u\right]. \tag{46}$$

$$\int_0^\infty e^{i|x|^{\frac{3}{2}}(u-1)^2}\mathrm{d}u \approx \int_{-\infty}^\infty e^{i|x|^{\frac{3}{2}}v^2}\mathrm{d}v = \sqrt{\pi}\,|x|^{-\frac{3}{4}}e^{i\frac{\pi}{4}}. \tag{47}$$

Therefore,

$$\mathrm{Ai}(x) = \frac{1}{\pi}\mathrm{Re}\left[\sqrt{\pi}\,|x|^{-\frac{1}{4}}e^{-i\frac{2}{3}|x|^{\frac{3}{2}}}e^{i\frac{\pi}{4}}\right] = \frac{1}{\sqrt{\pi}}|x|^{-\frac{1}{4}}\cos\left(\frac{2}{3}|x|^{\frac{3}{2}} - \frac{\pi}{4}\right). \tag{48}$$

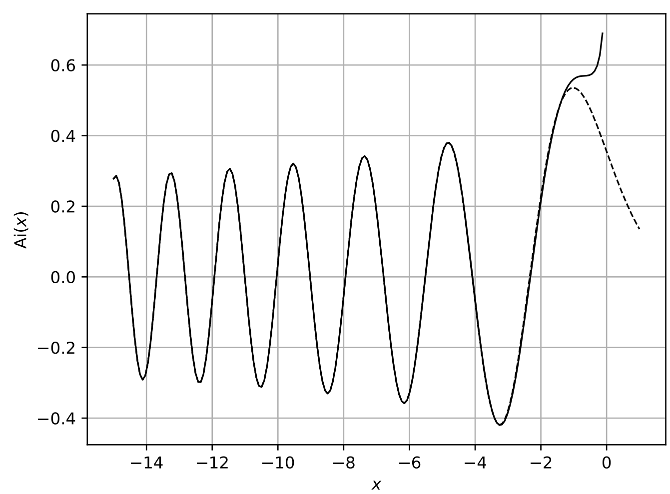


Figure 5: Asymptotics Eq. (48) (solid line) compared to numerically evaluated Eq. (36) (dashed line) for $-15 \le x \le 0$.

2 Integration by parts

If $\psi(t)$ in the integral Eq. (2) has no stationary point, $\psi'(t) = 0$, in the integration range $[a, b]$, the method of stationary phase is not applicable. In this case a simple integration by parts gives the leading asymptotic behavior.

$$\begin{aligned} I(x) &= \int_a^b f(t) e^{ix\psi(t)}\,\mathrm{d}t = \frac{1}{ix}\int_a^b \frac{f(t)}{\psi'(t)}\,\mathrm{d}\left(e^{ix\psi(t)}\right) \\ &= \frac{1}{ix}\frac{f(t)}{\psi'(t)}\,e^{ix\psi(t)}\bigg|_a^b - \frac{1}{ix}\int_a^b \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{f(t)}{\psi'(t)}\right) e^{ix\psi(t)}\,\mathrm{d}t. \end{aligned} \tag{49}$$

The integral on the right vanishes more rapidly than $1/x$ (Riemann–Lebesgue lemma). Therefore,

$$I(x) \sim \frac{1}{ix}\frac{f(t)}{\psi'(t)}\,e^{ix\psi(t)}\bigg|_a^b \tag{50}$$

as $x \to \infty$.

Example 1.

$$I(x) = \int_0^1 \frac{\cos(xt)}{1+t}\,\mathrm{d}t = \mathrm{Re}\int_0^1 \frac{e^{ixt}}{1+t}\,\mathrm{d}t. \tag{51}$$

Integrating the last integral by parts, we obtain

$$\int_0^1 \frac{e^{ixt}}{1+t}\,\mathrm{d}t = \frac{1}{ix}\int_0^1 \frac{1}{1+t}\,\mathrm{d}\left(e^{ixt}\right) = \frac{1}{ix}\left(\frac{e^{ix}}{2} - 1\right) + \frac{1}{ix}\int_0^1 \frac{e^{ixt}}{(1+t)^2}\,\mathrm{d}t. \tag{52}$$

The last term on the right is $\sim x^{-2}$ (see below), therefore the leading term in the approximation of Eq. (51) when $x \to \infty$ is

$$I(x) \approx \mathrm{Re}\left\{\frac{1}{ix}\left(\frac{e^{ix}}{2} - 1\right)\right\} = \frac{\sin(x)}{2x}. \tag{53}$$

We can continue the integration by parts of the integral in the right hand side of Eq. (52):

$$\int_0^1 \frac{e^{ixt}}{(1+t)^2}\,\mathrm{d}t = \frac{1}{ix}\int_0^1 \frac{1}{(1+t)^2}\,\mathrm{d}\left(e^{ixt}\right) = \frac{1}{ix}\left(\frac{e^{ix}}{4} - 1\right) + \frac{2}{ix}\int_0^1 \frac{e^{ixt}}{(1+t)^3}\,\mathrm{d}t. \tag{54}$$

Thus,

$$\int_0^1 \frac{e^{ixt}}{1+t}\,\mathrm{d}t = \frac{1}{ix}\left(\frac{e^{ix}}{2} - 1\right) - \frac{1}{x^2}\left(\frac{e^{ix}}{4} - 1\right) - \frac{2}{x^2}\int_0^1 \frac{e^{ixt}}{(1+t)^3}\,\mathrm{d}t. \tag{55}$$

The last term in the right hand side of Eq. (55) is of order x^{-3} and can be neglected, therefore

$$I(x) \approx \mathrm{Re}\left\{\frac{1}{ix}\left(\frac{e^{ix}}{2}-1\right)-\frac{1}{x^2}\left(\frac{e^{ix}}{4}-1\right)\right\} = \frac{\sin(x)}{2x}-\frac{1}{x^2}\left(\frac{\cos(x)}{4}-1\right) \tag{56}$$

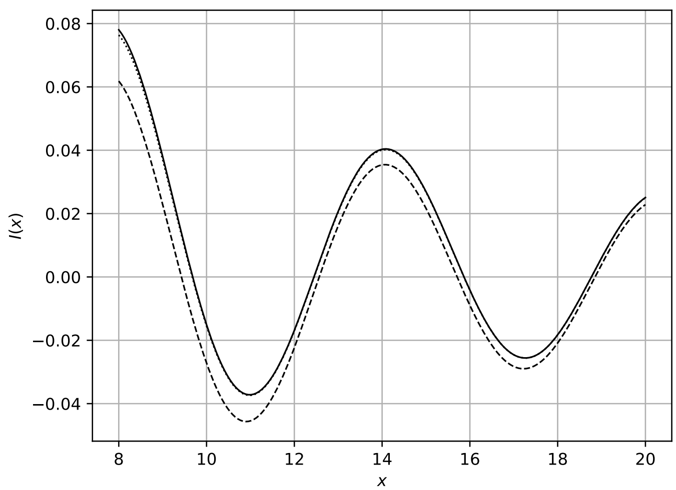


Figure 6: Asymptotics Eq. (53) (dashed line) and Eq. (56) (solid line) compared to numerically evaluated Eq. (51) (dotted line) for $8 \le x \le 20$.

Integration by parts can be a powerful tool even if a stationary point of the integrand is in the integration range but the contribution tho the integral from the integral end points is not small.

Example 2.

$$I(x) = \int_0^1 \cos(xt^2)\,\mathrm{d}t. \tag{57}$$

The main term in the asymptotics as $x \to \infty$ is due to the stationary point at $t = 0$.

$$I(x) \approx \mathrm{Re}\int_0^\infty e^{ixt^2}\,\mathrm{d}t = \frac{1}{2}\sqrt{\frac{\pi}{2x}}. \tag{58}$$

The approximation Eq. (58) is compared to the numerically evaluated integral Eq. (57) in Fig. 7. Although the numerical values of the approximation are close to the exact calculations, the important qualitative feature – small oscillations – is missing.

To do better, let's rewrite Eq. (57) as following:

$$I(x) = \mathrm{Re}\left[\int_0^\infty e^{ixt^2}\,\mathrm{d}t - \int_1^\infty e^{ixt^2}\,\mathrm{d}t\right]. \tag{59}$$

The first integral in Eq. (59) is exactly the main term of the stationary phase approximation Eq. (58). After integrating by parts in the second integral in Eq. (58), we obtain:

$$\mathrm{Re}\int_1^\infty e^{ixt^2}\,\mathrm{d}t = \mathrm{Re}\int_1^\infty \frac{\mathrm{d}\left(e^{ixt^2}\right)}{2ixt} \approx \mathrm{Re}\,\frac{e^{ixt^2}}{2ixt}\Bigg|_{t=1}^{t=\infty} = -\frac{\sin x}{2x}. \tag{60}$$

Therefore,

$$I(x) \approx \frac{1}{2}\sqrt{\frac{\pi}{2x}} + \frac{\sin x}{2x}. \tag{61}$$

The approximation Eq. (61) is compared to the numerically evaluated integral Eq. (57) in Fig. 7.

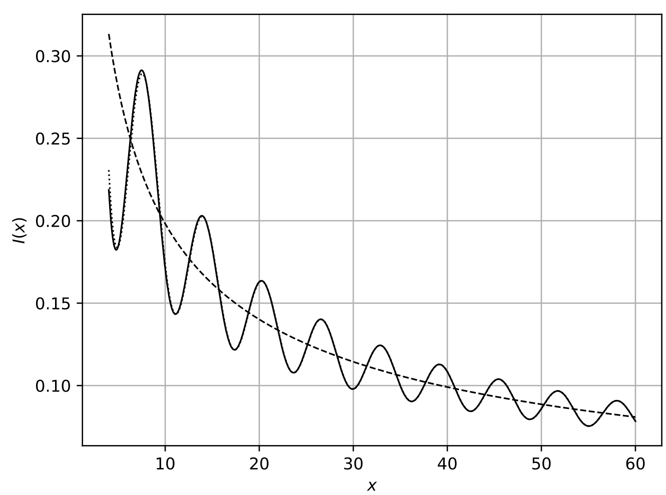


Figure 7: Asymptotics Eq. (58) (dashed line) and Eq. (61) (solid line) compared to the numerically evaluated integral Eq. (57) (dotted line) for $4 \le x \le 60$.

References

[1] Carl M. Bender and Steven A. Orszag. *Advanced Mathematical Methods for Scientists and Engineers*. Springer Verlag, 1999.

[2] Mark J. Ablowitz and Athanassios S. Fokas. *Complex Variables, Introduction and Applications*. 2nd ed. Cambridge Texts in Applied Mathematics. Cambridge University Press, 2003.