Liouville theorem

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The Liouville theorem states that if a function $f(z)$ is analytic for all finite z and is bounded at infinity then $f(z)$ is a constant.

Consider a circular contour, C_R, of radius R:

$$z = Re^{i\phi}, \quad 0 \le \phi < 2\pi, \quad \mathrm{d}z = iRe^{i\phi}\mathrm{d}\phi. \tag{1}$$

We take two arbitrary points, z_1 and z_2, inside the contour. The analyticity of $f(z)$ means that

$$f(z_1) = \frac{1}{2\pi i} \oint\limits_{z=Re^{i\phi}} \frac{f(z)\,\mathrm{d}z}{z - z_1} = \frac{R}{2\pi} \int\limits_0^{2\pi} \frac{f(z)\,\mathrm{d}\phi}{z - z_1} \tag{2}$$

and

$$f(z_2) = \frac{1}{2\pi i} \oint\limits_{z=Re^{i\phi}} \frac{f(z)\,\mathrm{d}z}{z - z_2} = \frac{R}{2\pi} \int\limits_0^{2\pi} \frac{f(z)\,\mathrm{d}\phi}{z - z_2}. \tag{3}$$

Here we keep using the notation $z = z(\phi) = Re^{i\phi}$ to simplify the expressions.

The difference between the values of the function $f(z)$ at z_1 and z_2 is as follows:

$$f(z_1) - f(z_2) = \frac{R}{2\pi} \int\limits_0^{2\pi} f(z) \left[\frac{1}{z - z_1} - \frac{1}{z - z_2} \right] \mathrm{d}\phi = \frac{R}{2\pi} \int\limits_0^{2\pi} \frac{f(z)}{z} \left[\frac{1}{1 - \frac{z_1}{z}} - \frac{1}{1 - \frac{z_2}{z}} \right] \mathrm{d}\phi. \tag{4}$$

We can estimate the absolute value of this difference by taking the limit $R \to \infty$. Let M be the finite bound of $f(z)$:

$$|f(z)| \le M, \quad \text{as} \quad |z| \to \infty. \tag{5}$$

Then,

$$\left| f(z_1) - f(z_2) \right| = \frac{R}{2\pi} \left| \int_0^{2\pi} \frac{f(z)}{z} \left[\frac{1}{1-\frac{z_1}{z}} - \frac{1}{1-\frac{z_2}{z}} \right] \mathrm{d}\phi \right| \tag{6}$$

$$\leq \frac{R}{2\pi} \int_0^{2\pi} \left| \frac{f(z)}{z} \right| \left| \frac{1}{1-\frac{z_1}{z}} - \frac{1}{1-\frac{z_2}{z}} \right| \mathrm{d}\phi \tag{7}$$

$$\leq \frac{M}{2\pi} \int_0^{2\pi} \left| \frac{1}{1-\frac{z_1}{z}} - \frac{1}{1-\frac{z_2}{z}} \right| \mathrm{d}\phi. \tag{8}$$

Here we used the relations

$$\left| \int_a^b F(\phi)\, \mathrm{d}\phi \right| \leq \int_a^b \left| F(\phi) \right| \mathrm{d}\phi \tag{9}$$

and

$$\left| \frac{f(z)}{z} \right| = \frac{|f(z)|}{|z|} \leq \frac{M}{R}. \tag{10}$$

In the limit $R \to \infty$, $\left|\frac{z_{1,2}}{z}\right| \ll 1$ and $\frac{1}{1-\frac{z_{1,2}}{z}} \approx 1 + \frac{z_{1,2}}{z}$. Hence,

$$\lim_{R\to\infty} \left| f(z_1) - f(z_2) \right| \leq \frac{M}{2\pi} \lim_{R\to\infty} \int_0^{2\pi} \left| \frac{1}{1-\frac{z_1}{z}} - \frac{1}{1-\frac{z_2}{z}} \right| \mathrm{d}\phi \tag{11}$$

$$\approx \frac{M}{2\pi} \lim_{R\to\infty} \int_0^{2\pi} \left| \frac{z_1 - z_2}{z} \right| \mathrm{d}\phi \leq \lim_{R\to\infty} \left| z_1 - z_2 \right| \frac{M}{R} = 0. \tag{12}$$

Thus

$$f(z_1) = f(z_2) \tag{13}$$

for all z_1 and z_2, i.e.

$$f(z) = \mathrm{const} \tag{14}$$

References

[1] A. O. Gogolin. *Lectures on Complex Integration*. Ed. by Elena G. Tsitsishvili and Andreas Komnik. Undergraduate Lecture Notes in Physics. Springer, 2014.