Name: ______________________________

Date: ______________________________

Collaborators: ____

(Collaborators submit their individually written assignments together, in class, in person)

Homework rules:

Show your work and indicate your reasoning in order to receive credit.

Guessing answers is not a solution.

You can use calculators, Computer Algebra systems, or any programming in general **only if the problem statement requests that**.

To solve the problems, use the methods that are taught in this class. If a problem statement includes directions for solutions, following those directions is required.

Question:	1	2	3	4	Total
Points:	15	20	25	25	85
Score:					

Instructor/grader comments:

Perturbation theory

1. (15 points) Use regular perturbation theory to find the solutions to the following equation for $\theta \gg 1$:

$$\tan\theta = \frac{1}{\theta}.$$

To verify your solution, plot in the same figure the graphs of $\tan(\theta)$ and $1/\theta$ (i.e. the left and the right hand sides of the equation), as well as your results for solutions of the equation (which are the intersection points of the graphs) for $0 \le \theta \le 10$. Use a Computer Algebra System. For plotting you may use code similar to the following:

```
T[n_] := n*Pi + your expression here
Show[
 Plot[{Tan[th], 1/th}, {th, 0, 10}, GridLines->Automatic, Frame->True],
 ListPlot[{{T[1],1/T[1]},{T[2],1/T[2]},{T[3],1/T[3]}},PlotMarkers->Automatic]
]
```

Attach a printout of your working CAS session.

The expected graph is shown in Fig. 1.

Hints/directions:

When θ is large, the graphs of $\tan(\theta)$ and $1/\theta$ intersect near $\theta = n\pi$, where n is a positive integer. Thus,

$$\theta_n = n\pi + x_n,$$

where x_n is small. Notice that $\tan\theta_n = \tan(n\pi + x_n) = \tan x_n$, therefore, the nonlinear equation takes the following form:

$$\tan x_n = \frac{1}{n\pi + x_n}.$$

Furthermore, for small values of its argument, s,

$$\tan(s) \approx s.$$

Hence, the original transcendental equation simplifies as follows:

$$x_n = \frac{1}{\frac{1}{\epsilon} + x_n},$$

where

$$\epsilon \equiv \frac{1}{n\pi}$$

is a small parameter of the problem.

Search for the solution in the following form

$$x_n \approx \epsilon a_1 + \epsilon^3 a_3,$$

Solve for a_1 and a_3. When simplifying the right hand side of the equation, consistently drop the terms of order 4 and higher in ϵ:

$$\frac{1}{\frac{1}{\epsilon} + \epsilon a_1 + \epsilon^3 a_3} = \frac{\epsilon}{1 + \epsilon^2 a_1 + \epsilon^4 a_3} \approx \frac{\epsilon}{1 + \epsilon^2 a_1}, \approx \epsilon \left(1 - \epsilon^2 a_1\right).$$

Present your final expressions for x_n and θ_n.

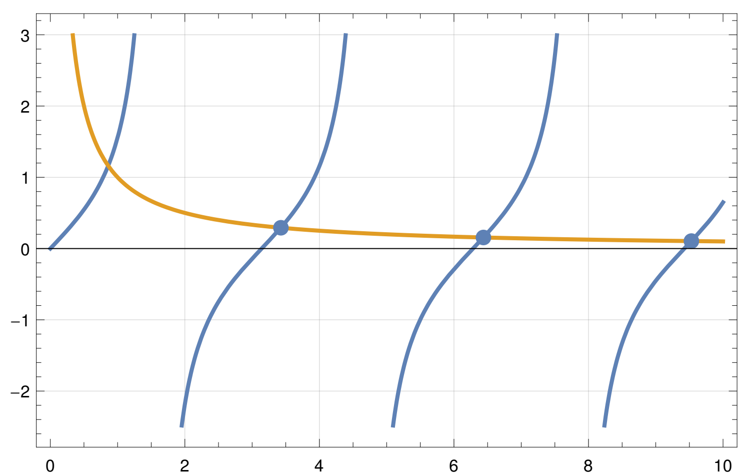

Figure 1: Expected result in Problem 1.

2. (20 points) Obtain a perturbative solution to the following initial value problem:

$$\frac{d^2y}{dx^2} = \sin(x)\,y, \quad y(0) = 1, \quad y'(0) = 1.$$

Find the solution of the unperturbed problem, $y_0(x)$, and its first correction, $y_1(x)$.

To verify your solution, plot in the same figure the graphs of numerical solution of the initial value problem, the graph of $y_0(x)$, and the graph of $y(x) = y_0(x) + y_1(x)$ for $0 \le x \le 1.5$. Use a Computer Algebra System.

For plotting you may use code similar to the following:

```
sol = DSolve[{y''[x]-Sin[x]*y[x]==0, y[0]==1, y'[0]==1}, y[x], {x, 0, 1}]
yexact[x_] := y[x] /. sol[[1]]
y0[x_] := your expression here
y1[x_] := your expression here
Plot[{yexact[x], y0[x], y0[x] + y1[x]}, {x, 0, 1},
   Frame->True, GridLines->Automatic, PlotLegends->"Expressions"]
```

Attach a printout of your working CAS session.

The expected graph is shown in Fig. 2.

Hints/directions:

Introduce a perturbative parameter, ϵ, to the problem:

$$\frac{d^2y}{dx^2} = \epsilon\,\sin(x)\,y, \quad y(0) = 1, \quad y'(0) = 1.$$

and search for the solution in the form

$$y(x) = y_0(x) + \epsilon\,y_1(x) + \dots,$$

where

$$y_0(0) = 1, \quad y_0'(0) = 1, \quad y_1(0) = 0, \quad y_1'(0) = 0, \quad \dots$$

For reference, the general solution of the differential equation

$$\frac{d^2z}{dx^2} = \sin(x)\,(x+1)$$

is as follows:

$$z(x) = -x\sin(x) - 2\cos(x) - \sin(x) + C_1x + C_2.$$

You still need to determine the integration constants C_1 and C_2 from the corresponding initial conditions.

Once you found $y_0(x)$ and $y_1(x)$, take $\epsilon = 1$.

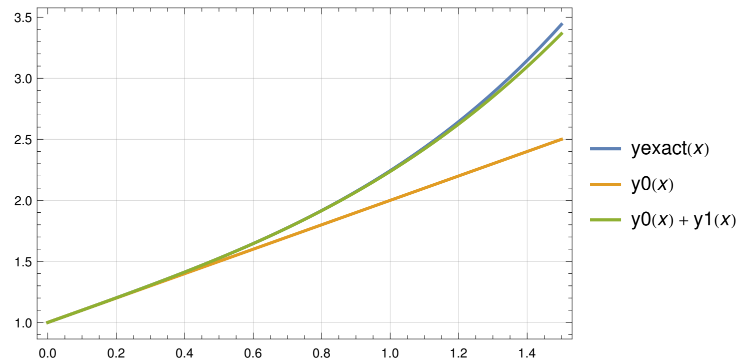

Figure 2: Expected result in Problem 2 (colors online).

3. (25 points) Find leading-order uniform approximations to the solutions of the following boundary value problem for $0 < \epsilon \ll 1$:

$$\epsilon\, y'' + \cosh(x)\, y' - y = 0, \quad y(0) = y(1) = 1, \quad 0 \le x \le 1.$$

Assume that the boundary layer is at $x = 0$.

To verify your solution, plot in the same figure the graphs of numerical solution of the initial value problem, the graph of the inner solution, the graph of the outer solution, and the graph of the uniform approximation for $\epsilon = 1/10$ and $0 \le x \le 1$. Use a Computer Algebra System.

You may use a code similar to the following:

```
eps = 1/10
eq = eps*y''[x] + Cosh[x]*y'[x] - y[x] == 0
sol = NDSolve[{eq, y[0] == 1, y[1] == 1}, y, {x, 0, 1}]
ynumer[x_] := y[x] /. sol
youter[x_] := your code here
yinner[x_] := your code here
yuniform[x_] := your code here
Plot[{ynumer[x], youter[x], yinner[x], yuniform[x]}, {x, 0, 1},
   Frame->True, GridLines -> Automatic, PlotLegends->"Expressions"]
```

Attach a printout of your working CAS session.

The expected graph is shown in Fig. 3.

Hints/directions:

For reference,

$$\int \frac{\mathrm{d}x}{\cosh(x)} = 2\arctan(e^x).$$

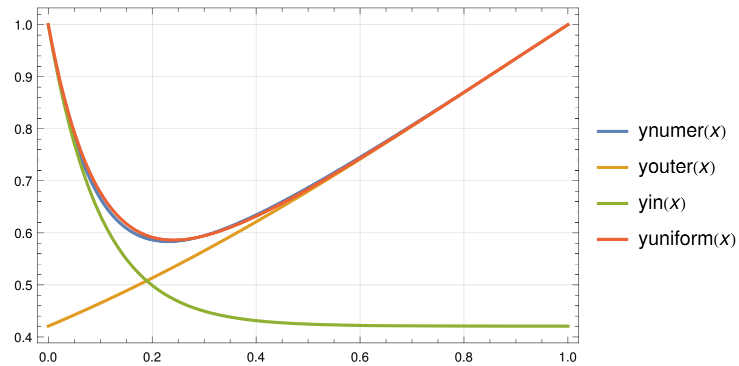

Figure 3: Expected result in Problem 3.

The method of averaging

4. (25 points) Find the approximate solution, $x(t)$ of the following initial value problem that describes a harmonic oscillator with nonlinear friction:

$$\ddot{x} + \epsilon\dot{x}^5 + x = 0, \quad x(0) = x_0, \quad \dot{x}(0) = 0,$$

where ϵ is a small positive parameter, $\dot{x}$ and $\ddot{x}$ denote the first and second derivatives of$x$ with respect to t. Compare your analytic approximation with the numerical solution of the differential equation.

To find the solution, you'll need to evaluate the following integral:

$$\overline{\sin^6(u)} \equiv \frac{1}{2\pi}\int_0^{2\pi} \sin^6(u)\mathrm{d}u.$$

To do so, first use the periodicity of the integrand to reduce the integration range from $[0, 2\pi]$ to $[0, \pi/2]$. Next, change the integration variable from u to $v = \sin^2 u$:

$$v = \sin^2 u, \quad \cos u = (1-v)^{\frac{1}{2}}, \quad \mathrm{d}v = 2\sin u\cos u\,\mathrm{d}u = 2v^{\frac{1}{2}}(1-v)^{\frac{1}{2}}\mathrm{d}u.$$

Reduce the integral to the Beta function, and evaluate it using the relation between the Beta and the Gamma function. For reference,

$$\Gamma(4) = 3! = 6, \quad \Gamma\left(\frac{7}{2}\right) = \frac{15}{8}\sqrt{\pi}.$$

To verify your solution, plot in the same figure the graphs of numerical solution of the initial value problem and your approximate solution for $\epsilon = 1/5$ and $0 \le t \le 25$. Use a Computer Algebra System.

You may use a code similar to the following:

```
eps = 1/5
xaver[t_] = your code here
eq = x''[t] + eps * (x'[t])^5 + x[t] == 0
sol = NDSolve[{eq, x[0]==1, x'[0]==0}, x, {t, 0, 25}]
xnumer[t_]:=x[t] /. sol
Plot[{xnumer[t], xaver[t]}, {t,0,25}, Frame->True,
   GridLines->Automatic, PlotLegends->"Expressions"]
```

Attach a printout of your working CAS session.

The expected graph is shown in Fig. 4.

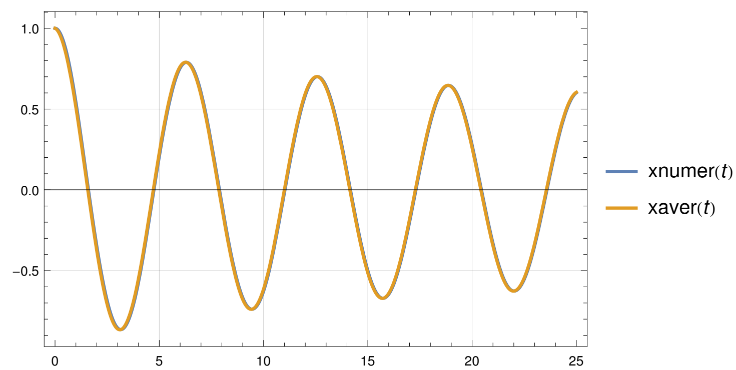

Figure 4: Expected result in Problem 4.