Name: ______________________________

Date: _______________________________

Collaborators: ____

(Collaborators submit their individually written assignments together, in class, in person)

Homework rules:

Show your work and indicate your reasoning in order to receive credit.

Guessing answers is not a solution.

You can use calculators, Computer Algebra systems, or any programming in general **only if the problem statement requests that**.

To solve the problems, use the methods that are taught in this class. If a problem statement includes directions for solutions, following those directions is required.

Question:	1	2	3	4	Total
Points:	15	15	30	15	75
Score:					

Instructor/grader comments:

The method of stationary phase

1. (15 points) As you found in Problem 2, HW06, the solution of the initial value problem,

$$x\,\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} + y = 0, \quad y(0) = 0, \quad y'(0) = 1,$$

can be written in the following integral form:

$$y(x) = \frac{\sqrt{x}}{\pi}\mathrm{Re}\left[\int_0^{\pi} e^{2i\sqrt{x}\sin\phi} e^{-i\phi}\mathrm{d}\phi\right].$$

Use the method of stationary phase to find the leading term of the asymptotics of $y(x)$ for $x \to \infty$.

Describe the location of the stationary point of the integrand, and the approximation you used for the integrand in its vicinity.

To verify your solution, plot on the same graph, for $0 \le x \le 150$ the numerical solution of the initial value problem and your asymptotics. Use a computer algebra system. Attach a printout of your CAS session.

The expected graph is shown in Fig. 1.

Hints:

For Mathematica, you may use the following code:

```
ode1 = {x * y''[x] + y[x]==0, y[0]==0, y'[0]==1}
sol = DSolve[ode1, y, {x, 0, 150}];
statphase[x_] = your code here
Plot[{y[x] /. sol, statphase[x]}, {x, 0, 150}, Frame->True,
   GridLines->Automatic, PlotLegends->{"numerical", "stationary phase"},
   PlotStyle->{Thick, Dashed}]
```

The Taylor series of $\sin\phi$ about $\phi = \frac{\pi}{2}$ is as follows:

$$\sin\phi \approx 1 - \frac{1}{2}\left(\phi - \frac{\pi}{2}\right)^2.$$

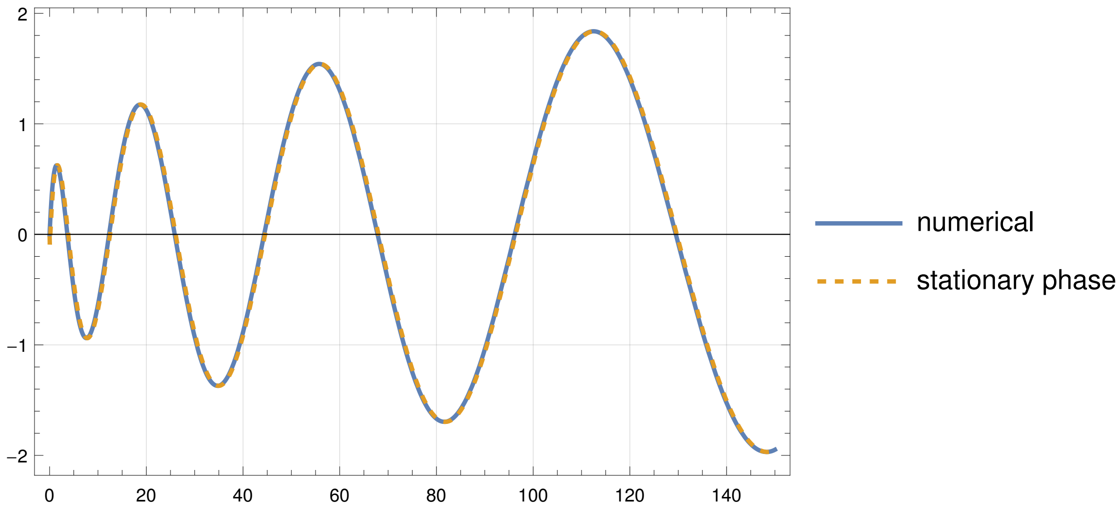

Figure 1: Expected result in Problem 1 (solid line – numerical solution, dashed line – asymptotics, color online).

2. (15 points) Find the leading term of the asymptotics of the following integral for $\lambda \to \infty$:

$$I(\lambda) = \int_0^\infty \cos\left(\lambda x^2 - x\right) \mathrm{d}x.$$

Describe the location of the stationary point of the integrand, and the approximation you used for the integrand in its vicinity.

To verify your solution, plot on the same graph, for $16 \le \lambda \le 1024$ the numerical value of the integral and your approximation. Use the logarithmic x-axis. Use a Computer Algebra System. Attach a printout of your CAS session.

The expected graph is shown in Fig. 2.

Hints: For Mathematica, you may use the following code:

```
fun[lambda_] = Integrate[Cos[lambda*x^2 - x], {x, 0, Infinity}]
statphase[lambda_] = your code here
LogLinearPlot[{fun[lambda], statphase[lambda]}, {lambda, 16, 1024},
         Frame->True, GridLines->Automatic, PlotStyle->{Thick, Dashed},
         PlotLegends->{"numerical", "stationary phase"}]
```

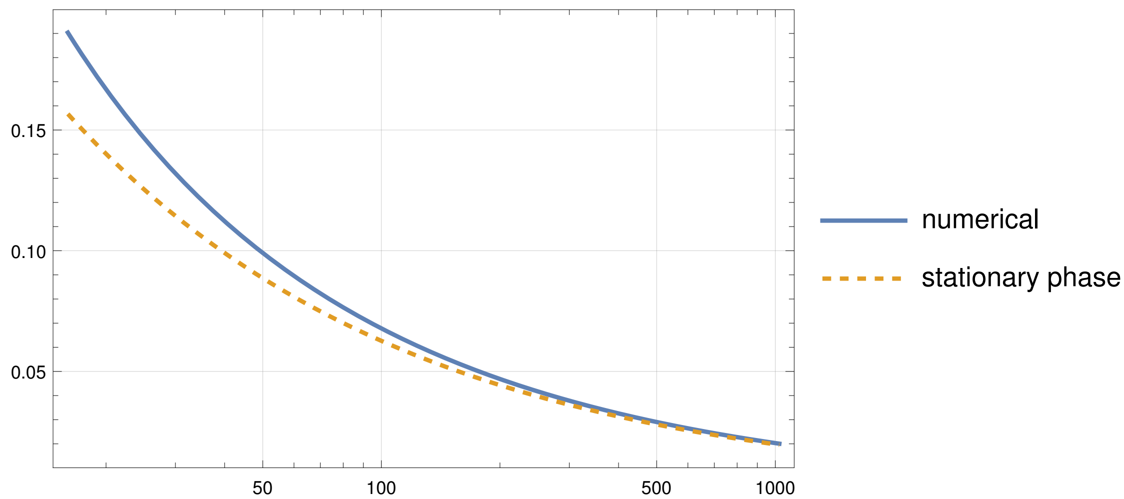

Figure 2: Expected result in Problem 2 (solid line – numerical solution, dashed line – asymptotics, color online).

3. (30 points) Use the method of stationary phase to find the leading term of the asymptotics of the following integral as $x \to \infty$:

$$I(\lambda) = \int_0^\infty \cos\left[\lambda\left(\frac{x^4}{2} - x^2\right)\right] \mathrm{d}x.$$

Describe the location(s) of the stationary point(s) of the integrand, and the approximation you used for the integrand.

To verify your solution, plot on the same graph, for $10 \le x \le 50$ the numerically evaluated integral and your asymptotics. Use a computer algebra system. Attach a printout of your CAS session.

The expected graph is shown in Fig. 3.

Hints: For Mathematica, you may use the following code:

```
fun[lambda_] = Integrate[Cos[lambda*(x^4/2 - x^2)], {x, 0, Infinity}]
statphase[lambda_] = your code here
Plot[{fun[lambda], statphase[lambda]}, {lambda, 10, 50}, Frame->True,
   GridLines->Automatic, PlotLegends->{"numerical", "stationary phase"},
   PlotStyle->{Thick, Dashed}]
```

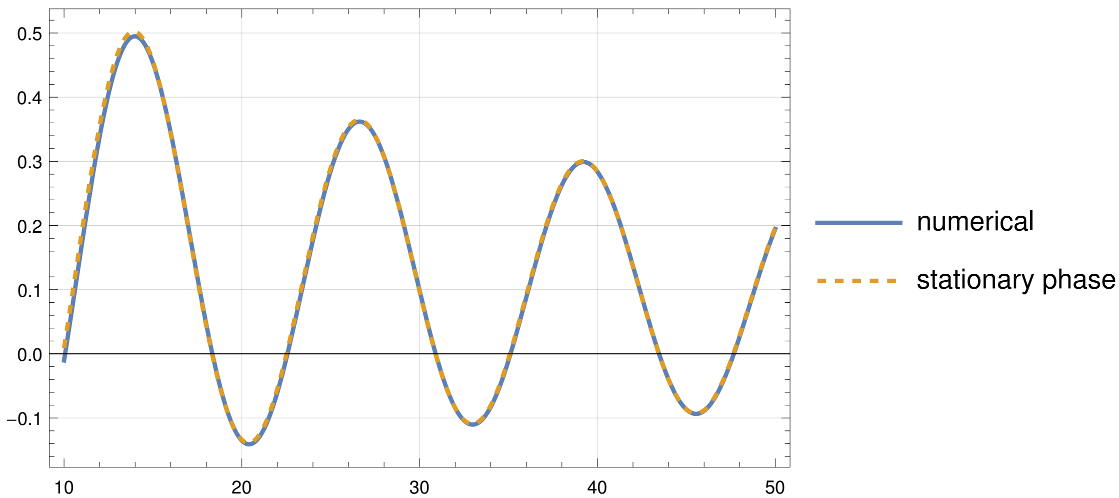

Figure 3: Expected result in Problem 3 (solid line – numerical solution, dashed line – asymptotics, color online).

A plot of $\phi(x) = \frac{x^4}{2} - x^2$ and of the integrand of the problem, $\cos(\lambda\phi(x))$, are shown in Fig. 4. Notice the positions of stationary points and corresponding behavior of the integrand.

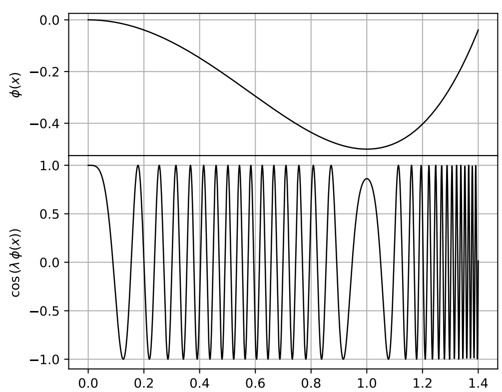

Figure 4: Integrand in Problem 3: $\phi(x)$ (top), and $\cos(\lambda\phi(x))$ for $\lambda = 200$ (bottom).

4. (15 points) Use the method of stationary phase to find the leading term of the asymptotics of the following integral as $\lambda \to \infty$:

$$I(\lambda) = \int_0^1 \cos\left(\lambda u^3\right) \sin(\pi u)\, \mathrm{d}u$$

Describe the location(s) of the stationary point(s) of the integrand, and the approximation you used for the integrand.

To verify your solution, plot on the same graph, for $10 \le \lambda \le 200$ the numerically evaluated integral and your asymptotics. Use a computer algebra system. Attach a printout of your CAS session.

The expected graph is shown in Fig. 5.

Hint: Recall that

$$\int_0^\infty u^q\, e^{i\lambda\, u^p}\, \mathrm{d}u = \frac{1}{p} e^{i\frac{(q+1)\pi}{2p}} \lambda^{-\frac{q+1}{p}}\, \Gamma\left(\frac{q+1}{p}\right).$$

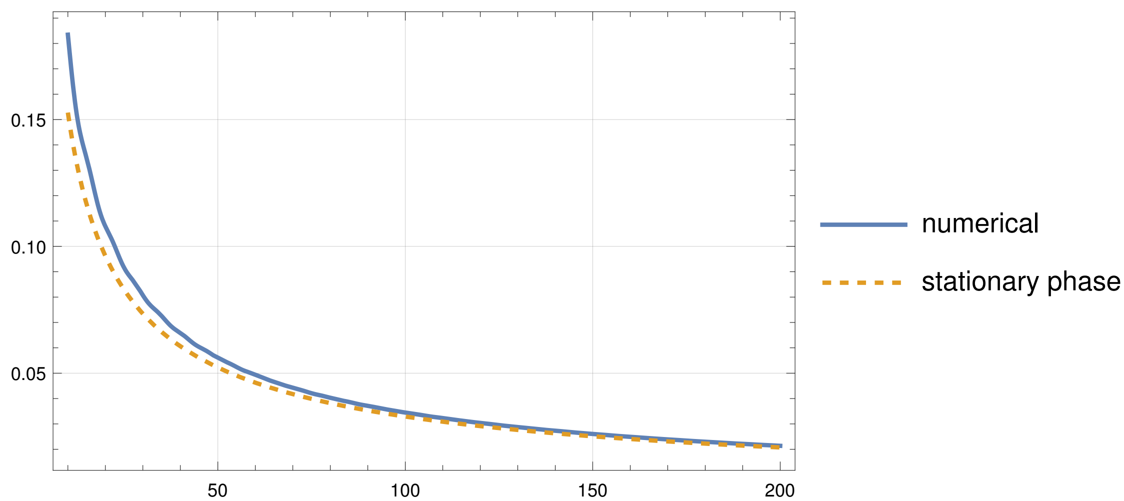

Figure 5: Expected result in Problem 4 (solid line – numerical solution, dashed line – asymptotics, color online).