Name: _____

Date: _____

Collaborators: _____

(Collaborators submit their individually written assignments together, in class, in person)

Homework rules:

Show your work and indicate your reasoning in order to receive credit.

Guessing answers is not a solution.

You can use calculators, Computer Algebra systems, or any programming in general **only if the problem statement requests that**.

To solve the problems, use the methods that are taught in this class. If a problem statement includes directions for solutions, following those directions is required.

Question:	1	2	3	4	5	Total
Points:	20	10	25	10	15	80
Score:						

Instructor/grader comments:

Laplace method for integrals

1. (20 points) Find the leading term of the asymptotics of the following integral for $\lambda \to \infty$:

$$I(\lambda) = \int_0^{\frac{\pi}{2}} \frac{\sqrt[3]{\cos(x)}}{(1+x^4)^\lambda}\, \mathrm{d}x.$$

Clearly describe the location of the maximum of the integrand, and the approximation you used for the integrand in the vicinity of the maximum. Evaluate the resulting integral in therms of the Gamma function.

To verify your result, use a Computer Algebra System to plot on the same graph the numerical value of the integral and your approximation for $10 < \lambda < 100$. Attach a printout of your CAS session.

The expected graph is shown in Fig. 1.

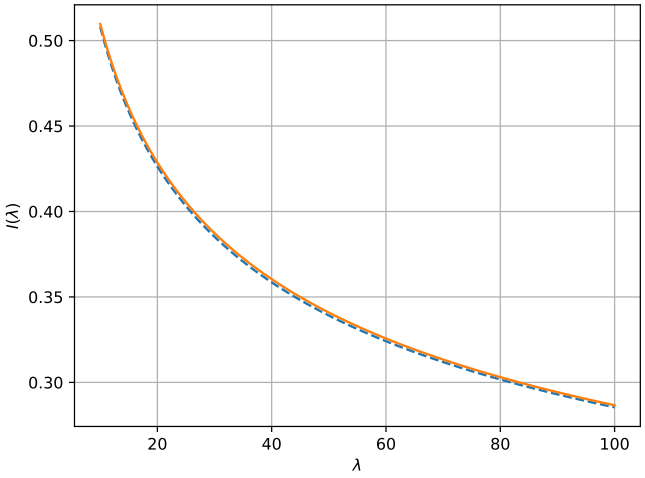

Figure 1: Expected result in Problem 1 (solid line – asymptotics, dashed line – numerically evaluated integral; colors online).

2. (10 points) Find the leading term of the asymptotics of the following integral for $\lambda \to \infty$:

$$I(\lambda) = \int_{-3}^{2} e^{-\lambda \tanh^2 x}\, \mathrm{d}x.$$

Clearly describe the location of the maximum of the integrand, and the approximation you used for the integrand in the vicinity of the maximum.

To verify your solution, use a Computer Algebra System to plot on the same graph the numerical value of the integral and your approximation for $10 < \lambda < 100$. Attach a printout of your CAS session.

The expected graph is shown in Fig. 2.

Hint: for $|x| \ll 1$, $\tanh(x) \approx x$; for $x \to \infty$, $\tanh(x) \approx 1$; for $x \to -\infty$, $\tanh(x) \approx -1$.

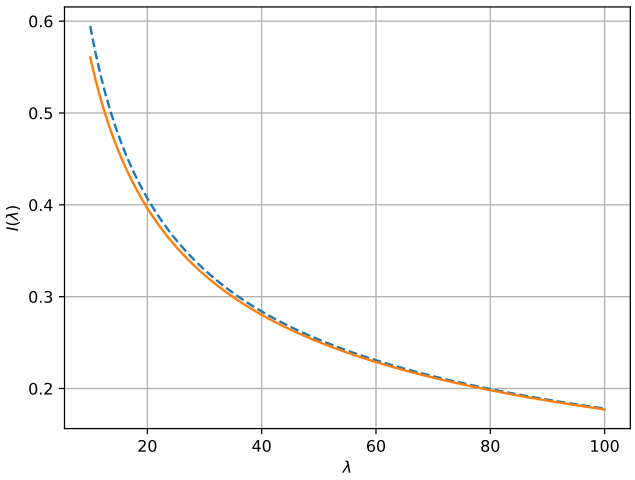

Figure 2: Expected result in Problem 2 (solid line – asymptotics, dashed line – numerically evaluated integral; colors online).

3. (25 points) Find the leading term of the asymptotics of the following integral for $\lambda \to \infty$:

$$I(\lambda) = \int_0^\infty e^{-\lambda x - \frac{4}{x^2}} \, \mathrm{d}x.$$

Hint: the integrand in this problem has a *moving maximum*. You need to change the integration variable so that the maximum of the integrand occurs at a fixed point independent on λ.

To verify your solution, use a Computer Algebra System to plot on the same graph the numerical value of the integral and your approximation for $3 < \lambda < 7$. Use log scale for y axis. Attach a printout of your CAS session.

The expected graph is shown in Fig. 3.

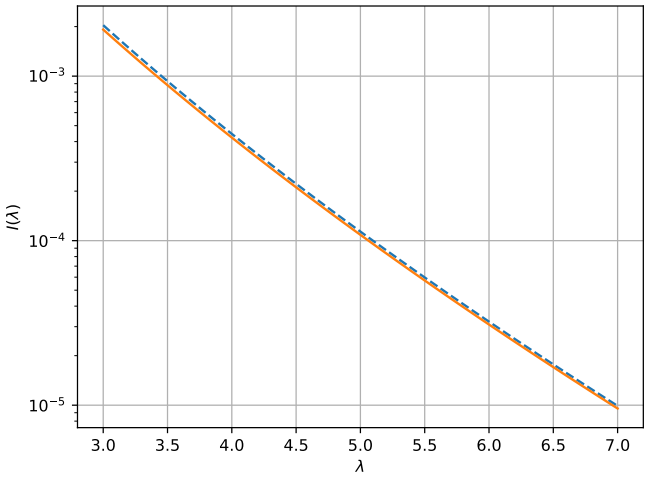

Figure 3: Expected result in Problem 3.

4. (10 points) Find the leading term of the asymptotics of the following integral for $\lambda \to \infty$:

$$I(\lambda) = \int_0^1 \cos\left(\lambda x^3\right) \mathrm{d}x.$$

Describe the location of the stationary point of the integrand, and the approximation you used for the integrand in its vicinity.

To verify your solution, use a Computer Algebra System to plot on the same graph the numerical value of the integral and your approximation for $3 \le \lambda \le 60$. **Do not** attach a printout of your CAS session.

The expected graph is shown in Fig. 4.

5. (15 points) Improve the approximation for the integral you considered in Problem 4 by evaluating the first correction term.

   To verify your solution, use a Computer Algebra System to plot on the same graph the numerical value of the integral, your approximation from Problem 4, and your improved approximation for $3 \le \lambda \le 60$. Attach a printout of your CAS session.

   The expected graph is shown in Fig. 4.

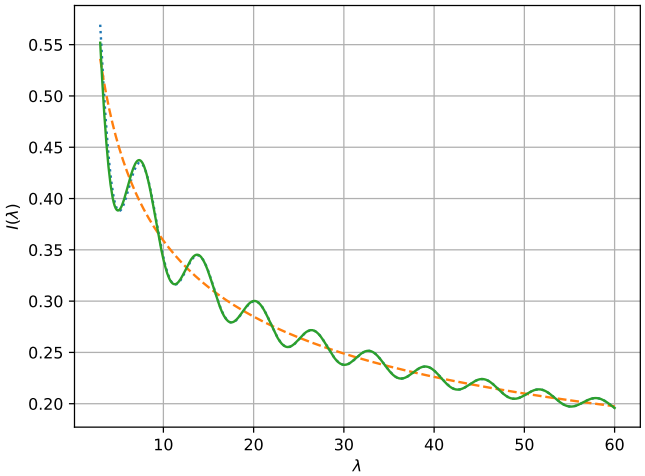

Figure 4: Expected result in Problems 4 and 5 (dashed line – the leading term of the asymptotics, solid line – the leading term with the first correction, dotted line – numerically evaluated integral).