Name: ______________________________

Date: ______________________________

Collaborators: ____

(Collaborators submit their individually written assignments together, in class, in person)

Homework rules:

Show your work and indicate your reasoning in order to receive credit.

Guessing answers is not a solution.

You can use calculators, Computer Algebra systems, or any programming in general **only if the problem statement requests that**.

To solve the problems, use the methods that are taught in this class. If a problem statement includes directions for solutions, following those directions is required.

Question:	1	2	Total
Points:	20	50	70
Score:			

Instructor/grader comments:

Integral to stump a computer algebra system

1. (20 points) Construct a definite integral that you can evaluate analytically but a computer algebra cannot. Use the method described in the handout "The integral that stumped Feynman". **Do not use the integrands similar to ones discussed in the handout.** Use a computer algebra system for finding the real and the imaginary parts of your complex expressions.

   To verify your result, numerically evaluate your integral and your answer. Use a computer algebra system for numerics. Enclose a printout of you computer algebra session.

   An example of expected outcome (from the handout):

   The function $f(z)$ below is analytic on the unit disk.

$$f(z) = \exp\left(\frac{2+z}{3+z}\right).$$

   The process described in the handout leads to the construction of the following integrand and the integral identity:

$$\int_0^{2\pi} \exp\left(\frac{7+5\cos\phi}{10+6\cos\phi}\right)\cos\left(\frac{\sin\phi}{10+6\cos\phi}\right)\mathrm{d}\phi = 2\pi e^{2/3}.$$

Laplace method for ODEs

2. Use the Laplace's method for differential equations to solve the following initial value problem for $x \geq 0$:

$$x\,\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} + y = 0, \quad y(0) = 0, \quad y'(0) = 1 \ .$$

   (a) (15 points) Use the Laplace's method to find the integral solution containing one integration constant. Use a CCW complex closed contour around the origin as the integration contour.

   (b) (15 points) Use your integral representation of $y(x)$, to calculate $y(0)$ and $y'(0)$. Chose the integration constant to satisfy the initial conditions.

   Hint: Recall that

$$e^{-\frac{1}{z}} = 1 - \frac{1}{1!\,z} + \frac{1}{2!\,z^2} - \frac{1}{3!\,z^3} + \dots$$

   Use the power series above to obtain the Laurent series for the integrands for $y(0)$ and $y'(0)$.

(c) (15 points) Rewrite your integral solution to the form suitable for its numerical evaluation.

Let t be the integration variable in your expression for $y(x)$. Introduce a new integration variable, $u = \sqrt{x}t$. Deform the integration contour to a unit circle in the complex plane.

Make another change of integration variable, $u = e^{i\phi}$, $0 \le \phi \le 2\pi$. Since your solution must be a real function of x, replace the integrand with its real part.

Hint: recall that $\cos(\alpha) = \frac{1}{2}\left(e^{i\alpha} + e^{-i\alpha}\right)$, $\sin(\alpha) = \frac{1}{2i}\left(e^{i\alpha} - e^{-i\alpha}\right)$, $\mathrm{Re}\left(e^{i\alpha}\right) = \cos(\alpha)$.

(d) (5 points) To verify your solution, plot on the same graph your integral solution (in the form obtained in Part c) and the solution of the original initial value problem. Use a computer algebra system. Attach a printout of your CAS session.

Hint: For Mathematica, you may use the following code:

```
ode1 = {x*y''[x]+y[x]==0,y[0]==0,y'[0]==1}
sol = DSolve[ode1, y, {x,0,50}];
lapsol[x_] = <your pre-integral factor> * Integrate[your integrand here,
  {phi, 0, 2*Pi}];
Plot[{lapsol[x], y[x]/. sol}, {x,0,50}, Frame->True,
  GridLines->Automatic, PlotLegends->{"analytic","numerical"}]
```

The expected graph is shown in Fig. 1.

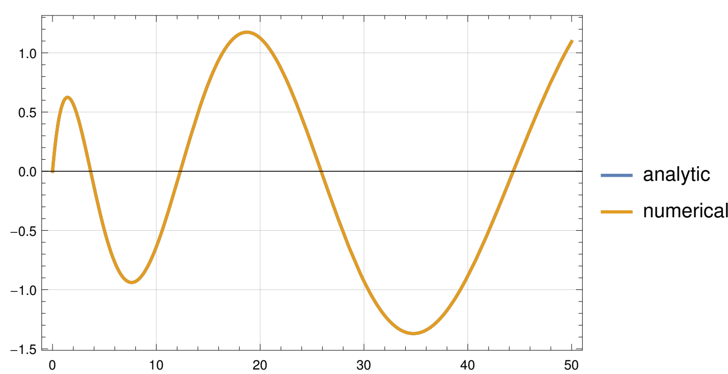

Figure 1: Expected result in Problem 2: two graphs overlap.