Name: ______________________________

Date: ______________________________

Collaborators: ____

(Collaborators submit their individually written assignments together and in person)

Question:	1	2	3	4	5	6	7	Total
Points:	15	10	10	10	10	10	10	75
Score:								

Instructor/grader comments:

Euler's formula

1. (15 points) Evaluate the following sum as a closed form expression:

$$S(\theta, n) = \sin(\theta) + \sin(2\theta) + \ldots + \sin(n\theta) = \sum_{k=1}^{n} \sin(k\theta). \qquad (A)$$

Your final expression must be valid for all $n > 1$ and contain only trigonometric functions and possibly constants. As a quick check, make sure that in the limit $\theta \to 0$ you get $S(0, n) = 0$.

   1. To verify your result, plot on the same graph, for $n = 6$ and for $-\pi \leq \theta \leq \pi$, the direct sum (A) and your answer. The resulting graph should look similar to Figure 1.
   2. Print your Mathematica session and attach the printout in the back of your homework booklet.

Hints:

   1. Recall that
$$\sin\theta = \mathrm{Im}\left(e^{i\theta}\right).$$
   2. The sum of imaginary parts equals to the imaginary part of the sum.
   3. Recall the expression for the sum of the geometric series:
$$a + a^2 + a^3 + \ldots + a^n = \sum_{k=1}^{n} a^k = \frac{a\,(1 - a^n)}{1 - a}.$$
   4. Rewrite the expressions $1 - e^{i\psi}$ in the numerator and the denominator of your formula for the sum as follows:
$$1 - e^{i\psi} = e^{i\psi/2}\left(e^{-i\psi/2} - e^{i\psi/2}\right) = -2ie^{i\psi/2}\sin\left(\frac{\psi}{2}\right).$$
   5. Your Mathematica code could be something like the following:

```
f[t_, n_] := NSum[Sin[k*t], {k, 1, n}]
g[t_, n_] := your_answer_here
nn = 6
Plot[{f[t, nn], g[t, nn]}, {t, -Pi, Pi}, Frame->True,
 GridLines->Automatic, PlotLegends -> "Expressions"]
```

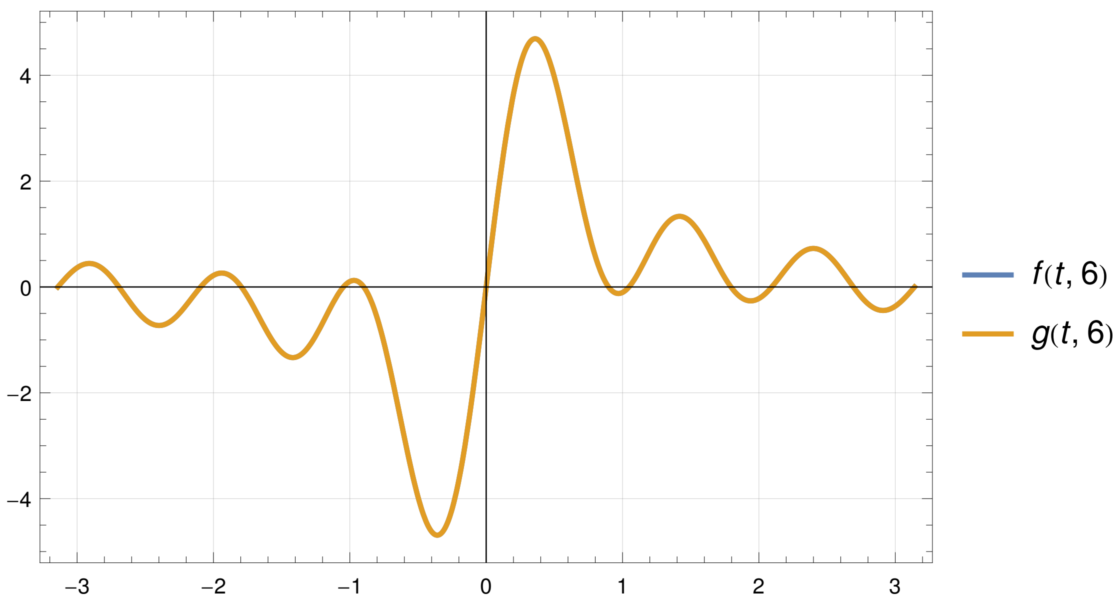

Figure 1: Expected graph in Problem 1 (colors online)

The problems 2–6 below are not calculator, CAS, or computer problems. No credits will be given if those tools are used for solutions.

Gamma and Beta functions

2. (10 points) Evaluate the integral in terms of Gamma function. Simplify the expression as much as possible.

$$I = \int_0^\infty e^{-x^4} \mathrm{d}x$$

Answer: $I = \Gamma\left(\frac{5}{4}\right)$

3. (10 points) Evaluate the following expressions. Here $\Gamma()$ and $B()$ are the Euler gamma and beta functions respectively. Assume that only the values of $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$ and $\Gamma(1) = 1$ are known.

   (a) $\Gamma\left(\frac{3}{2}\right)$

   (b) $\Gamma\left(-\frac{3}{2}\right)$

   (c) $B\left(-\frac{1}{2}, \frac{5}{2}\right)$

Complex numbers

4. (10 points) Find the coordinate and the polar form of the following complex number:

$$Z = \left(\frac{\sqrt{2} - i\sqrt{2}}{1 + i\sqrt{3}}\right)^{26}.$$

Answer: $Z = -e^{-i\frac{\pi}{6}} = -\frac{\sqrt{3}}{2} + \frac{i}{2}$

5. (10 points) Find the absolute value of the following complex number:

$$Z = \left(\frac{3}{5} + i\frac{4}{5}\right)^n \left(1 + i\sqrt{3}\right)^4$$

for arbitrary positive integers n.

Answer: $|Z| = 16$.

6. (10 points) Find the values of $Z = \left(\sqrt{i}\right)^i$.

Answer: $Z = e^{-\frac{\pi}{4} - \pi n}$, where $n = 0, \pm 1, \pm 2, \ldots$

7. (10 points) Find the coordinate and the polar forms of the solutions of the equation:

$$z^4 = \sqrt{3} - i.$$

Answer: the solutions of the equation are sketched in Fig. 2.

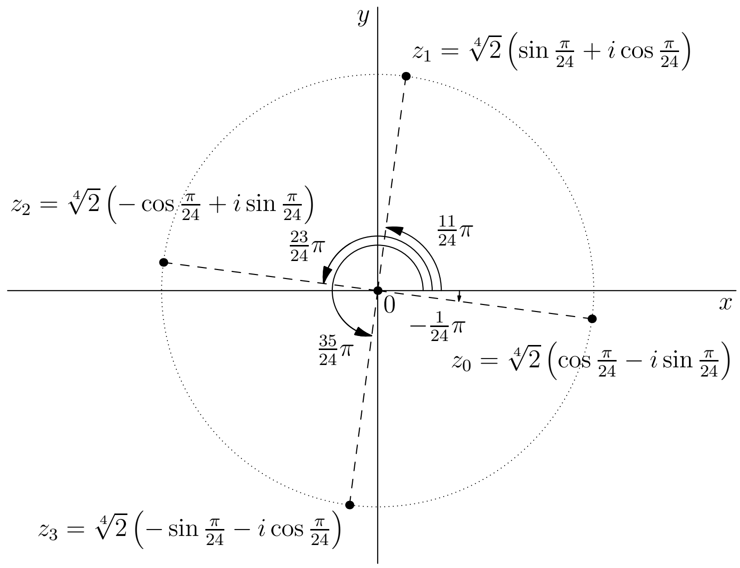

Figure 2: Roots of the equation $z^4 = \sqrt{3} - i$.