Name: ______________________________

Date: ______________________________

Collaborators: ____

(Collaborators submit their individually written assignments together, in class, in person)

Question:	1	2	3	Total
Points:	20	30	20	70
Score:				

Instructor/grader comments:

Leibniz' rule

1. (20 points) Find the positive value of x that maximizes the value of the following integral

$$I(x) = \int_{x-\frac{1}{2}}^{x+\frac{1}{2}} \frac{\mathrm{d}u}{\Gamma(u)}, \qquad (B)$$

where $\Gamma(u)$ is the Gamma function.

For reference, the integrand in Eq. (B) is sketched in Figure 1.

Present your solution as an **algebraic expression**, i.e. as an expression built up from constants, variables, and the algebraic operations (addition, subtraction, multiplication, division and exponentiation by an exponent that is a rational number).

To verify that your answer makes sense, draw the vertical line $u = x$ in the graph Figure 1.

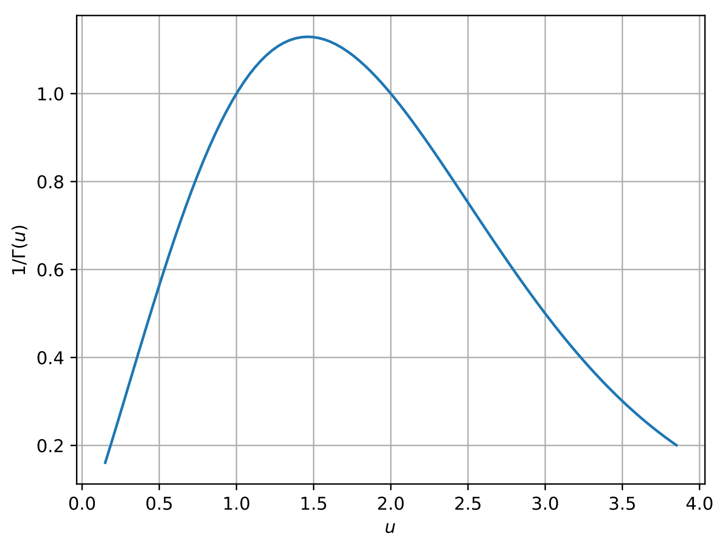

Figure 1: Graph of $1/\Gamma(u)$.

Hints: Recall that the derivative of a function at its maximum is zero. Find the derivative of $I(x)$. Simplify the equation that you obtained using the relations

$$\Gamma(z+1) = z\,\Gamma(z),$$

where z can be an arbitrary expression. Solve the resulting equation for x.

2. (30 points) Find the solution of the following integral equation:

$$f(x) = \int_0^\infty e^{-|x-s|} f(s)\,\mathrm{d}s, \quad 0 \le x < \infty. \qquad (C)$$

Equation (C) is a linear integral equation, therefore its solution can be determined up to an arbitrary multiplication constant. Fix that constant by requiring that

$$f(0) = 1. \qquad (D)$$

Note that combining (C) and (D) we get an additional relation $\int_0^\infty e^{-s} f(s)\,\mathrm{d}s = 1$.

The recommended steps to the solution are as follows:

a. Rewrite the integral in the right hand side of Eq. (C) as the sum of two integrals – one for $0 \le s \le x$ and another for $x \le s < \infty$. In each of the integrals replace $|x - s|$ with an expression that does not use the absolute values.

b. Differentiate the "new" form of the integral equation. Simplify the expression. Use it to find the value of $f'(0)$.

c. Differentiate the first derivative of the integral equation once more. Simplify the second derivative of the equation to reduce it to the second order ordinary differential equation.

   Hint: you should get the familiar equation for harmonic oscillations.

d. Write down the general solution of the differential equation. Determine the integration constants using the values of $f(0)$ that is give and $f'(0)$ that you already found.

e. Use Mathematica to verify your solution by substituting it back to the integral equation (in its "new" form).

   Hint: the Mathematica code could be something like the following:

```
eq = f[x] - Integrate[your_first_integrand, {s, 0, x}] -
            Integrate[your_second_integrand, {s, x, Infinity}]
f[x_] := your_solution
eq // Simplify
```

Print your Mathematica session and attach the printout to the rest of your homework.

Gamma and Beta functions

3. (20 points) The following integral appears in the expression for the collapse time of cavitation bubbles:

$$\int_0^1 \frac{u^4\,\mathrm{d}u}{\sqrt{1-u^6}}.$$

Evaluate the integral in terms of beta function. (Hint: introduce a new integration variable $v = u^6$.) Rewrite your answer in terms of gamma functions with arguments less than one. (Hint: use the relation $\Gamma(x+1) = x\,\Gamma(x)$ to reduce the values of the arguments, if needed.)

Answer: $\frac{\sqrt{\pi}}{2}\frac{\Gamma\left(\frac{5}{6}\right)}{\Gamma\left(\frac{1}{3}\right)}$.