EULER'S INTEGRALS

Spring semester 2025

https://www.phys.uconn.edu/~rozman/Courses/P2400_25S/

Last modified: January 29, 2025

1 The Gamma function

In the eighteenth century, Leonhard Euler (a Swiss-born mathematician and physicist, 1707-1783) concerned himself with the problem of interpolating between *factorials*,

$$n! \equiv n \cdot (n-1) \cdot (n-2) \cdot \ldots \cdot 2 \cdot 1 , \qquad (1)$$

that are defined for integer *n*. This problem led Euler in 1729 to the *gamma function*, a generalization of the factorial function that gives meaning to *x*! when *x* is any positive number. Furthermore, the Gamma function can be extended to negative numbers and as well as to complex numbers.

Consider the function $\Gamma(x)$ defines as following integral,

$$\Gamma(x) \equiv \int_{0}^{\infty} t^{x-1} e^{-t} dt$$
(2)

which is convergent for x - 1 > -1, i.e. x > 0. Integral Eq. (2) is known as the *Euler integral* of the second kind.

Using the relation $e^{-t} dt = -d(e^{-t})$ and integrating Eq. (2) by parts once, we get:

$$\Gamma(x) = -\int_{0}^{\infty} t^{x-1} d(e^{-t}) = -t^{x-1} e^{-t} \int_{0}^{\infty} e^{-t} d(t^{x-1})$$
$$= (x-1) \int_{0}^{\infty} t^{x-2} e^{-t} dt = (x-1)\Gamma(x-1),$$
(3)

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Repeatedly integrating by parts, we obtain,

$$\Gamma(x) = (x-1)\Gamma(x-1) = (x-1)(x-2)\Gamma(x-2) = (x-1)(x-2)(x-3)\Gamma(x-3) = \dots$$
(4)

If x = n is a positive integer, then repeating the integration by parts we eventually arrive to the expression:

$$\Gamma(n) = (n-1) \cdot (n-2) \cdot \ldots \cdot 2 \cdot \Gamma(1).$$
(5)

Noticing that

$$\Gamma(1) = \int_{0}^{\infty} e^{-t} dt = 1,$$
(6)

we conclude that for an positive integer argument n,

$$\Gamma(n) = (n-1)! \tag{7}$$

The gamma function has been used as a means of generalizing functions, operations, etc., which are commonly defined in terms of factorials. In addition, the gamma function is useful in the evaluation of many non-elementary integrals and in the definition of other special functions. A graph of $\Gamma(x)$ is shown in Fig. 1.



Figure 1: The gamma function. The (red) dots mark the values of n! for n = 1, 2, 3, 4 (color online).

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To obtain another useful integral representation of $\Gamma(x)$, let's set the integration variable in Eq. (2) to $t = u^2$, dt = 2udu, $0 \le u < \infty$. We get

$$\Gamma(x) = 2 \int_{0}^{\infty} u^{2x-1} e^{-u^2} du.$$
 (8)

In particular,

$$\Gamma\left(\frac{1}{2}\right) = 2\int_{0}^{\infty} e^{-u^{2}} du = \int_{-\infty}^{\infty} e^{-u^{2}} du.$$
(9)

The integral Eq. (9) is known as *Gaussian integral*.

Gamma function satisfies the following identity for all complex z:

$$\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin(\pi z)},\tag{10}$$

referred to as the Euler's reflection formula.

Another remarkable property of Gamma function is so-called Legendre duplication formula:

$$\Gamma(2z) = \frac{2^{2z-1}}{\sqrt{\pi}} \Gamma(z) \Gamma\left(z + \frac{1}{2}\right).$$
(11)

Although the definition of the gamma function – Eq. (2) – is only valid (on the real axis) for positive arguments, its domain can be extended with *analytic continuation* to negative arguments by shifting the negative argument to positive values by using either the Euler's reflection formula Eq. (10) or the fundamental property Eq. (3) written in the form,

$$\Gamma(x-1) = \frac{\Gamma(x)}{x-1}.$$
(12)

Example 1. Evaluate the integral:

$$I = \int_{0}^{\infty} x^4 e^{-x^3} \mathrm{d}x.$$
 (13)

Let $u = x^3$, then

$$x = u^{\frac{1}{3}}, \quad x^4 = u^{\frac{4}{3}}, \quad dx = \frac{1}{3}u^{-\frac{2}{3}}du, \quad 0 \le u < \infty.$$
 (14)

$$I = \int_{0}^{\infty} x^{4} e^{-x^{3}} dx = \frac{1}{3} \int_{0}^{\infty} u^{\frac{4}{3}} u^{-\frac{2}{3}} e^{-u} du = \frac{1}{3} \int_{0}^{\infty} u^{\frac{2}{3}} e^{-u} du = \frac{1}{3} \int_{0}^{\infty} u^{\frac{5}{3}-1} e^{-u} du = \frac{1}{3} \Gamma\left(\frac{5}{3}\right).$$
(15)

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Example 2. Fractional-order derivatives:

In addition to generalizing the notion of factorials, the gamma function can be used in a variety of situations to transform a discrete processes into a continuous one.

We can illustrate the concept of fractional derivatives by first recalling the derivative formula from calculus:

$$\frac{\mathrm{d}^{n}}{\mathrm{d}x^{n}}x^{a} = a(a-1)\cdots(a-n+1)x^{a-n}, \quad a \ge 0, \quad n = 1, 2, 3, \dots$$
(16)

In terms of the gamma function, we can rewrite Eq. (16) as follows:

$$\frac{\mathrm{d}^n}{\mathrm{d}x^n} x^a = \frac{\Gamma(a+1)}{\Gamma(a-n+1)} x^{a-n}.$$
(17)

The right-hand side of this expression is meaningful for any real number n for which $\Gamma(a - n + 1)$ is defined. Hence, we assume that the same is true of the left-hand side and write

$$\frac{\mathrm{d}^{\nu}}{\mathrm{d}x^{\nu}}x^{a} = \frac{\Gamma(a+1)}{\Gamma(a-\nu+1)}x^{a-\nu},\tag{18}$$

where ν is not restricted to integer values. Equation Eq. (18)) provides a method of computing *fractional-order derivatives* of polynomials.

2 The Beta function

Another function useful in various applications is the *beta function*, often called the *eulerian integral of the first kind*.

$$B(x,y) \equiv \int_{0}^{1} t^{x-1} (1-t)^{y-1} \mathrm{d}t.$$
(19)

If we make the change of variable u = 1 - t, we find

$$B(x,y) = \int_{0}^{1} (1-u)^{x-1} u^{y-1} dt.$$
 (20)

from which we deduce the symmetry property

$$B(x, y) = B(y, x).$$
⁽²¹⁾

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We obtain another representation of the Beta function if we make the following change of integration variable:

$$t = \sin^2 \theta, \quad 0 \le \theta \le \frac{\pi}{2}, \quad \mathrm{d}t = 2\cos\theta\sin\theta\,\mathrm{d}\theta,$$
 (22)

$$B(x,y) = 2 \int_{0}^{\frac{\pi}{2}} \sin^{2x-1}\theta \cos^{2y-1}\theta \,\mathrm{d}\theta.$$
 (23)

To establish the relation between Beta and Gamma functions, let's calculate the following product using the integral representation Eq. (8) for Gamma function:

$$\Gamma(x)\Gamma(y) = 4 \int_{0}^{\infty} u^{2x-1} e^{-u^{2}} du \int_{0}^{\infty} v^{2y-1} e^{-v^{2}} dv = 4 \iint_{0}^{\infty} u^{2x-1} v^{2y-1} e^{-(u^{2}+v^{2})} du dv.$$
(24)

The presence of the term $u^2 + v^2$ in the integrand suggests the change of variables from cartesian (u, v) to polar (r, θ) :

$$u = r\cos\theta, \quad v = r\sin\theta, \quad r^2 = u^2 + v^2, \quad 0 \le r < \infty, \quad 0 \le \theta \le \frac{\pi}{2}, \quad dudv \to r drd\theta.$$
 (25)

Thus,

$$\Gamma(x)\Gamma(y) = 4 \int_{0}^{\infty} r^{2(x+y)-1} e^{-r^2} dr \int_{0}^{\frac{\pi}{2}} \sin^{2x-1}\theta \cos^{2y-1}\theta d\theta = \Gamma(x+y)B(x,y).$$
(26)

Therefore,

$$B(x,y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}.$$
(27)

We now use Eq. (27) to calculate the value $\Gamma(\frac{1}{2})$. On the one hand, from Eq. (23), $B(\frac{1}{2}, \frac{1}{2}) = \pi$. On the other hand, from Eq. (27), $B(\frac{1}{2}, \frac{1}{2}) = \Gamma^2(\frac{1}{2})$. Therefore,

$$\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}.\tag{28}$$

Example 1. Find the area enclosed by the curve $x^4 + y^4 = 1$ (see Fig. 2).

$$y(x) = (1 - x^4)^{\frac{1}{4}}, \quad -1 \le x \le 1.$$
 (29)

$$A = 4 \int_{0}^{1} y(x) \,\mathrm{d}x.$$
 (30)

Let $u = x^4$, then

$$x = u^{\frac{1}{4}}, \quad \mathrm{d}x = \frac{1}{4}u^{-\frac{3}{4}}\mathrm{d}u, \quad 0 \le u \le 1.$$
 (31)

$$A = 4\frac{1}{4}\int_{0}^{1} u^{-\frac{3}{4}}(1-u)^{\frac{1}{4}} du = \int_{0}^{1} u^{\frac{1}{4}-1}(1-u)^{\frac{5}{4}-1} du = B\left(\frac{1}{4}, \frac{5}{4}\right) = \frac{\Gamma\left(\frac{1}{4}\right)\Gamma\left(\frac{5}{4}\right)}{\Gamma\left(\frac{3}{2}\right)}.$$
 (32)



Figure 2: The area enclosed by the curve $x^4 + y^4 = 1$ (color online).

We can simplify the answer Eq. (32) using the relations:

$$\Gamma\left(\frac{5}{4}\right) = \Gamma\left(\frac{1}{4}+1\right) = \frac{1}{4}\Gamma\left(\frac{1}{4}\right), \qquad \Gamma\left(\frac{3}{2}\right) = \Gamma\left(\frac{1}{2}+1\right) = \frac{1}{2}\Gamma\left(\frac{1}{2}\right) = \frac{\sqrt{\pi}}{2}.$$
 (33)

Thus,

$$A = \frac{\Gamma\left(\frac{1}{4}\right)\Gamma\left(\frac{5}{4}\right)}{\Gamma\left(\frac{3}{2}\right)} = \frac{\Gamma\left(\frac{1}{4}\right)\cdot\frac{1}{4}\Gamma\left(\frac{1}{4}\right)}{\frac{\sqrt{\pi}}{2}} = \frac{\Gamma^{2}\left(\frac{1}{4}\right)}{2\sqrt{\pi}}.$$
(34)

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