# DIFFERENTIATING UNDER THE INTEGRAL SIGN

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https://www.phys.uconn.edu/~rozman/Courses/P2400_25S/

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The *Leibniz integral rule*, named after Gottfried Wilhelm Leibniz, states that for a definite integral where the integrand and the integration limits are differentiable functions of a parameter $t$, its derivative with respect to the parameter can be determined as follows:

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\int_{a(t)}^{b(t)} f(x,t)\,\mathrm{d}x\right) = \int_{a(t)}^{b(t)} \left(\frac{\partial}{\partial t} f(x,t)\right)\mathrm{d}x + f\left(b(t)\right)\frac{\mathrm{d}b}{\mathrm{d}t} - f\left(a(t)\right)\frac{\mathrm{d}a}{\mathrm{d}t} \tag{1}$$

## 1 The case of the integrand depending on the parameter

Let

$$I(t) = \int_a^b f(x,t)\,\mathrm{d}x, \tag{2}$$

where $a$, $b$ are fixed parameters.

Then,

$$\frac{\mathrm{d}I}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t}\left(\int_a^b f(x,t)\,\mathrm{d}x\right) = \int_a^b \left(\frac{\partial}{\partial t} f(x,t)\right)\mathrm{d}x, \tag{3}$$

Indeed, considering the definition of the derivative as the limit,

$$\frac{\mathrm{d}I}{\mathrm{d}t} = \lim_{\Delta t \to 0} \frac{I(t+\Delta t) - I(t)}{\Delta t}, \tag{4}$$

and expanding $f(x, t+\Delta t)$ into Taylor series,

$$f(x, t+\Delta t) = f(x,t) + \frac{\partial f}{\partial t}\Delta t + O\left(\Delta t^2\right). \tag{5}$$

$$\begin{aligned} I(t+\Delta t) - I(t) &= \int_a^b f(x, t+\Delta t)\,\mathrm{d}x - \int_a^b f(x,t)\,\mathrm{d}x = \int_a^b \left[f(x,t+\Delta t) - f(x,t)\right]\mathrm{d}x \\ &= \int_a^b \left[\frac{\partial f}{\partial t}\Delta t + O\left(\Delta t^2\right)\right]\mathrm{d}x = \left[\int_a^b \left(\frac{\partial f}{\partial t}\right)\mathrm{d}x\right]\Delta t + O\left(\Delta t^2\right). \end{aligned} \tag{6}$$

Substituting Eq. (6) into Eq. (4), and taking the limit $\Delta t \to 0$, we obtain Eq. (3).

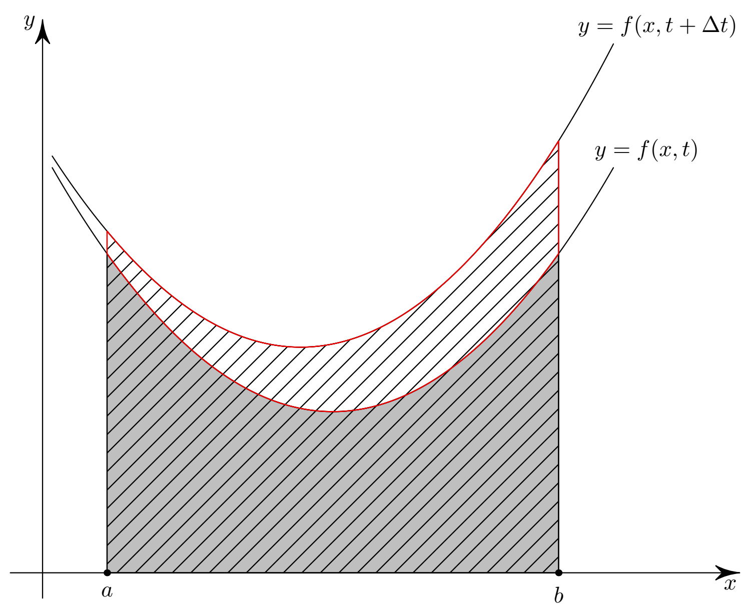


Figure 1: $I(t)$ (gray background), $I(t+\Delta t)$ (hatched background), and their difference in Eq. (6).

## 2 Case of the integration limits depending on the parameter

Let

$$I(t) = \int_{a(t)}^{b(t)} f(x)\,\mathrm{d}x. \tag{7}$$

where the integration limits $a(t)$ and $b(t)$ are functions of the parameter $t$ but the integrand $f(x)$ does not depend on $t$.

Then,

$$\frac{\mathrm{dI}}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t}\left(\int_{a(t)}^{b(t)} f(x)\,\mathrm{d}x\right) = f\left(b(t)\right)\frac{\mathrm{d}b}{\mathrm{d}t} - f\left(a(t)\right)\frac{\mathrm{d}a}{\mathrm{d}t}, \tag{8}$$

Indeed,

$$\frac{\mathrm{d}I}{\mathrm{d}t} = \lim_{\Delta t\to 0}\frac{I(t+\Delta t) - I(t)}{\Delta t} \tag{9}$$

$$I(t+\Delta t) = \int_{a(t+\Delta t)}^{b(t+\Delta t)} f(x)\,\mathrm{d}x. \tag{10}$$

$$\begin{aligned} I(t+\Delta t) - I(t) &= \int_{a(t+\Delta t)}^{b(t+\Delta t)} f(x)\,\mathrm{d}x - \int_{a(t)}^{b(t)} f(x)\,\mathrm{d}x = \int_{b(t)}^{b(t+\Delta t)} f(x)\,\mathrm{d}x - \int_{a(t)}^{a(t+\Delta t)} f(x)\,\mathrm{d}x \\ &= \left(b(t+\Delta t) - b(t)\right) f\left(b(t)\right) - \left(a(t+\Delta t) - a(t)\right) f\left(a(t)\right) + O\left(\Delta t^2\right) \\ &= \left[f\left(b(t)\right)\frac{\mathrm{d}b}{\mathrm{d}t} - f\left(a(t)\right)\frac{\mathrm{d}a}{\mathrm{d}t}\right]\Delta t + O\left(\Delta t^2\right), \end{aligned} \tag{11}$$

where we used that $a(t+\Delta t) - a(t) = \frac{\mathrm{d}a}{\mathrm{d}t}\Delta t + O\left(\Delta t^2\right)$ and similar for $b$.

Combining Eq. (11) and Eq. (9), and taking the limit $\Delta t \to 0$, we obtain Eq. (8).

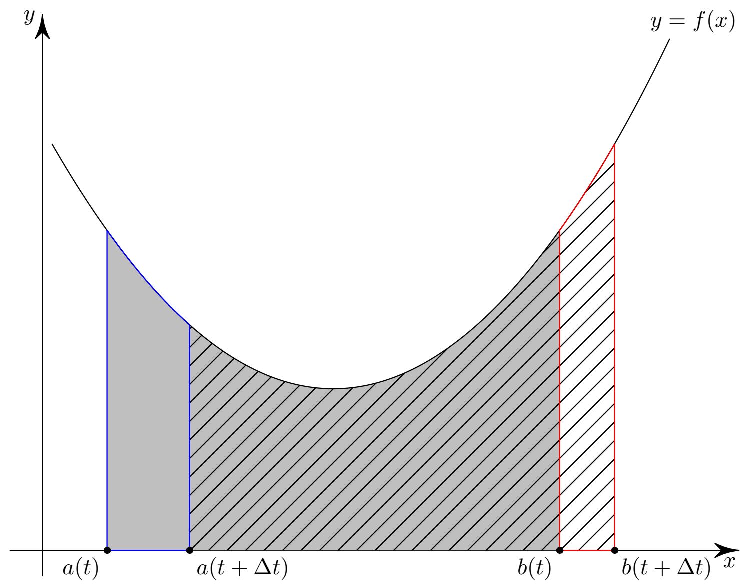


Figure 2: $I(t)$ (gray background), $I(t+\Delta t)$ (hatched background), and their difference in Eq. (11).

# 3 General case

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\int_{a(t)}^{b(t)} f(x,t)\,\mathrm{d}x\right) = \int_{a(t)}^{b(t)}\left(\frac{\partial}{\partial t} f(x,t)\right)\mathrm{d}x + f\left(b(t)\right)\frac{\mathrm{d}b}{\mathrm{d}t} - f\left(a(t)\right)\frac{\mathrm{d}a}{\mathrm{d}t} \tag{12}$$

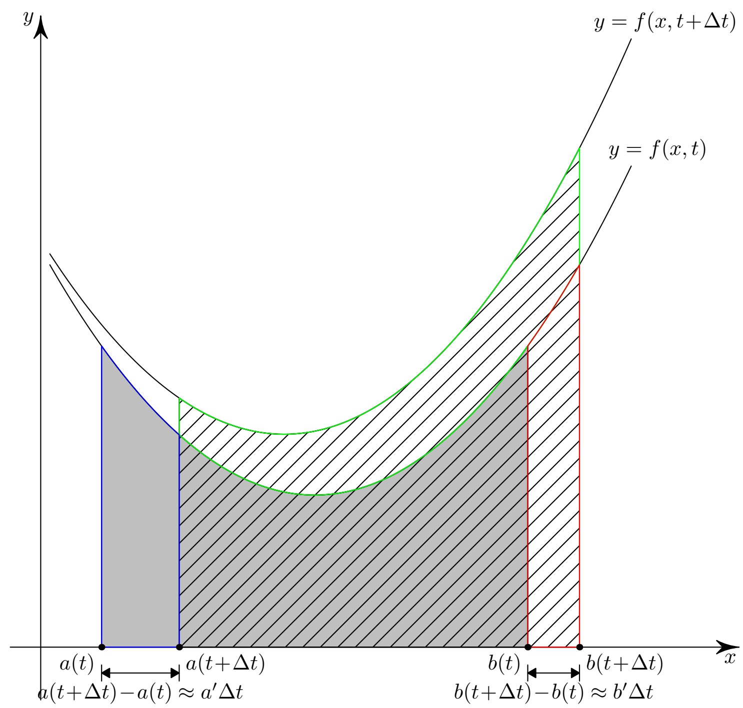


Figure 3: $I(t)$ (gray background), $I(t+\Delta t)$ (hatched background), and their difference.

# 4 Examples

**Problem 1.** Find the solution of the following integral equation:

$$\phi(x)+\frac{1}{2}\int_{-1}^{1}|x-s|\,\phi(s)\,\mathrm{d}s=x,\quad -1\leq x\leq 1. \tag{13}$$

**Solution:** we rewrite the integral term in the equation as follows,

$$\left(\hat{L}\phi\right)(x)\equiv\int_{-1}^{1}|x-s|\,\phi(s)\,\mathrm{d}s=\int_{-1}^{x}(x-s)\,\phi(s)\,\mathrm{d}s+\int_{x}^{1}(s-x)\,\phi(s)\,\mathrm{d}s,\quad -1\leq x\leq 1, \tag{14}$$

where in the first integral $x\geq s$ and $|x-s|=x-s$; in the second integral $x\leq s$ and $|x-s|=s-x$. The integral equation get the form:

$$\phi(x)+\frac{1}{2}\int_{-1}^{x}(x-s)\,\phi(s)\,\mathrm{d}s+\frac{1}{2}\int_{x}^{1}(s-x)\,\phi(s)\,\mathrm{d}s=x. \tag{15}$$

Taking the derivative of Eq. (15) with respect to $x$ and using the result Eq. (12), we obtain:

$$\phi'(x)+\frac{1}{2}\int_{-1}^{x}\phi(s)\,\mathrm{d}s-\frac{1}{2}\int_{x}^{1}\phi(s)\,\mathrm{d}s=1. \tag{16}$$

Taking the derivative of Eq. (16), we obtain the following ordinary differential equation:

$$\phi''(x)+\phi(x)=0. \tag{17}$$

The general solution of Eq. (17) is as follows:

$$\phi(x)=A\cos(x)+B\sin(x), \tag{18}$$

where $A$ and $B$ are the integration constants. To find them we plug the solution Eq. (18) back into the integral equation and set $x=0$:

$$\phi(0)=A, \tag{19}$$

$$\left(\hat{L}\phi\right)(0)=\frac{1}{2}\int_{-1}^{1}|0-s|\,\phi(s)\,\mathrm{d}s=\frac{A}{2}\int_{-1}^{1}|s|\cos(s)\,\mathrm{d}s+\cancelto{0}{\frac{B}{2}\int_{-1}^{1}|s|\sin(s)\,\mathrm{d}s}=\alpha A, \tag{20}$$

where

$$\alpha \equiv \int_0^1 s\cos(s)\,\mathrm{d}s. \tag{21}$$

$\alpha$ is positive, since the integrand is positive. (Also, $\alpha = \sin(1) + \cos(1) - 1$ but we do not need the exact value.)

The equation

$$A + \alpha A = 0, \tag{22}$$

where $\alpha > 0$ has the solution

$$A = 0. \tag{23}$$

Thus,

$$\phi(x) = B\sin(x). \tag{24}$$

To determine the constant $B$ we plug the solution Eq. (24) into Eq. (16) and set $x = 1$:

$$\phi'(x) = B\cos(x), \quad \phi'(1) = B\cos(1), \tag{25}$$

$$\frac{1}{2}\int_{-1}^{1}\phi(s)\,\mathrm{d}s = \frac{B}{2}\int_{-1}^{1}\sin(s)\,\mathrm{d}s = 0, \tag{26}$$

$$B\cos(1) = 1, \tag{27}$$

$$B = \frac{1}{\cos(1)}. \tag{28}$$

Therefore, the solution of Eq. (13) is as follows:

$$\phi(x) = \frac{\sin x}{\cos(1)}, \quad -1 \le x \le 1. \tag{29}$$