CAUCHY'S INTEGRAL THEOREM

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https://www.phys.uconn.edu/~rozman/Courses/P2400_25S/

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Cauchy's theorem states that if $f(z)$ is analytic at all points on and inside a closed contour C in the complex plane, then the integral of the function around that contour vanishes:

$$\oint_C f(z)\,\mathrm{d}z = 0. \tag{1}$$

Here is the proof of Cauchy's theorem following the presentation in [1, pp. 363-5], [2, pp. 5-6].

We assume that the contour C bounds a *star-shaped region* and that $f'(z)$ is bounded everywhere within and on C. The geometric concept of "star-shaped" is as following. A region is star-shaped if a point O can be found such that every ray from O intersects the bounding curve of the region in precisely one point. An example of such a region is shown in Fig. 1, left. A region which is not star-shaped is illustrated in Fig. 1, right. Restricting our proof to a star-shaped region is not a limitation on the theorem, since any simply connected region may be broken up into a number of star-shaped regions and the Cauchy theorem applied to each one.

Take the point O of the star-shaped region to be the origin of our reference frame. Define $F(\lambda)$ as follows:

$$F(\lambda) = \lambda \oint_C f(\lambda z)\,\mathrm{d}z, \tag{2}$$

where the real parameter $\lambda \in [0,1]$. When the variable z traverses the integration contour C, the argument of function f in Eq. (2), λz, traverses scaled contours (see Fig. 1, (e) and (f)). Only if C is a star-shaped contours, the scaled contours lie entirely inside C, i.e. inside the area of analiticity of function f.

The Cauchy theorem Eq. (1) states that

$$F(1) = 0. \tag{3}$$

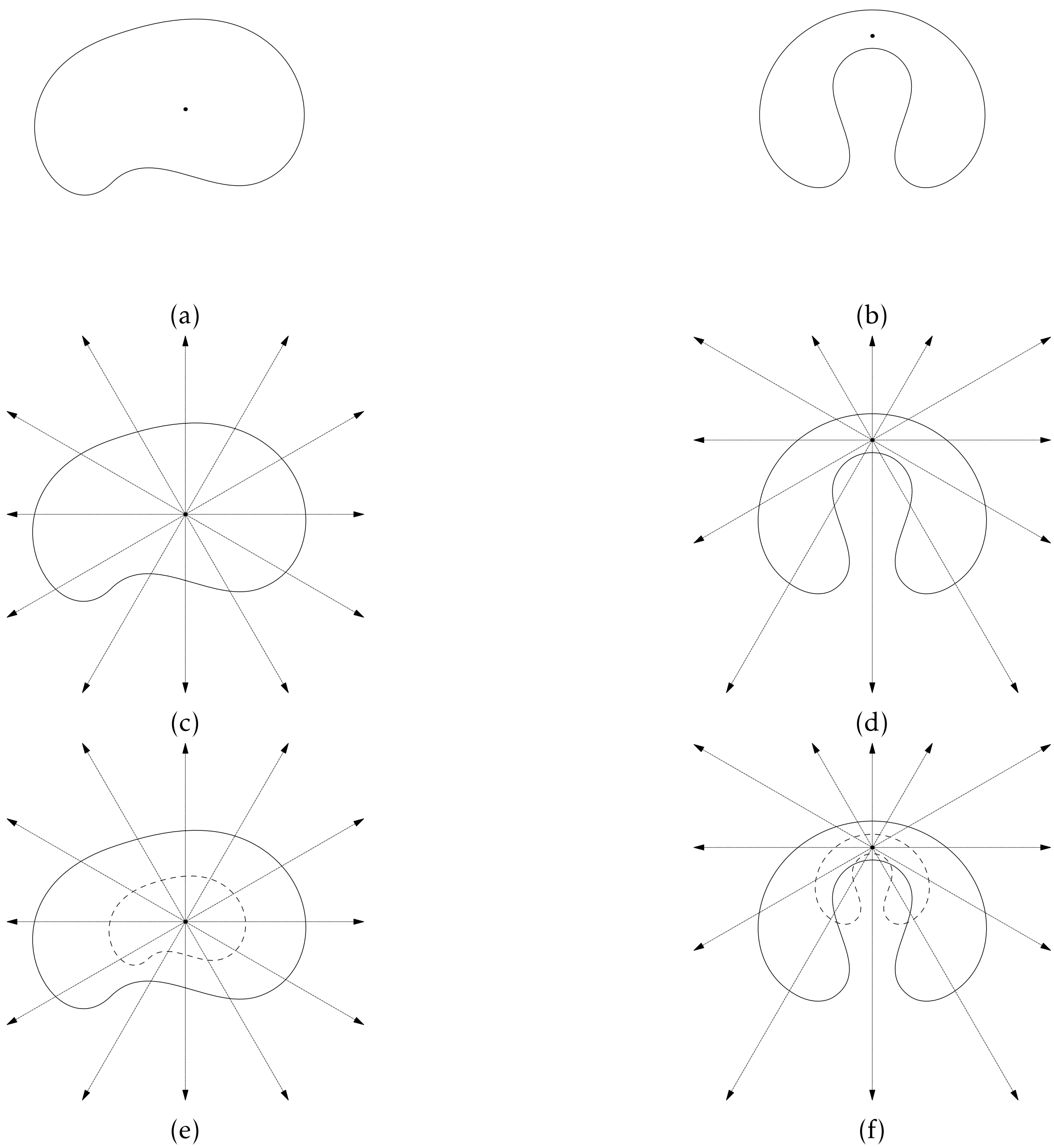

Figure 1: Star-shaped region (figures on the left) and and non-star-shaped region (on the right). Solid lines indicate integrating contours, dashed lines - contours scaled by the factor 0.5. Only star-shaped contours guaranteed to have the scaled contours inside the unscaled one.

To prove Eq. (3), we take derivative of $F(\lambda)$ with respect to λ:

$$\frac{\mathrm{d}F}{\mathrm{d}\lambda} = \oint_C f(\lambda z)\,\mathrm{d}z + \lambda \oint_C z f'(\lambda z)\,\mathrm{d}z = \oint_C f(\lambda z)\,\mathrm{d}z + \oint_C z\,\mathrm{d}f(\lambda z) \tag{4}$$

Integrate the second of these integrals by parts (which is possible only if $f'(z)$ is bounded):

$$\frac{\mathrm{d}F}{\mathrm{d}\lambda} = \oint_C f(\lambda z)\,\mathrm{d}z + [z f(\lambda z)] - \oint_C f(\lambda z)\,\mathrm{d}z = [z f(\lambda z)], \tag{5}$$

where the square brackets indicates that we take the difference of the values at the beginning and at the end of the contour. Since $z f(\lambda z)$ is a single-valued function, the expression in the square brackets vanishes for a closed contour so that

$$\frac{\mathrm{d}F}{\mathrm{d}\lambda} = 0 \quad \text{or} \quad F(\lambda) = \text{const.} \tag{6}$$

To evaluate the constant, we notice that letting $\lambda = 0$ in Eq. (2) yields $F(0) = 0$. Therefore $F(1) = 0$, i.e.

$$\oint_C f(z)\,\mathrm{d}z = 0. \tag{7}$$

which concludes the proof.

Deformation of contours

An immediate and important consequence of the Cauchy theorem is as follows: the value of a contour integral does not change if the contour is deformed within the analyticity domain of the integrand.

Specifically, if we have two different contours γ_{ab} and γ'_{ab} connecting two points a and b in the complex plane (see Fig. 2(a), then

$$\int_{\gamma_{ab}} f(z)\,\mathrm{d}z = \int_{\gamma'_{ab}} f(z)\,\mathrm{d}z. \tag{8}$$

Indeed, we have

$$\int_{\gamma_{ab}} f(z)\,\mathrm{d}z - \int_{\gamma'_{ab}} f(z)\,\mathrm{d}z = \int_{\gamma_{ab}+\gamma'_{ba}} f(z)\,\mathrm{d}z = 0 \tag{9}$$

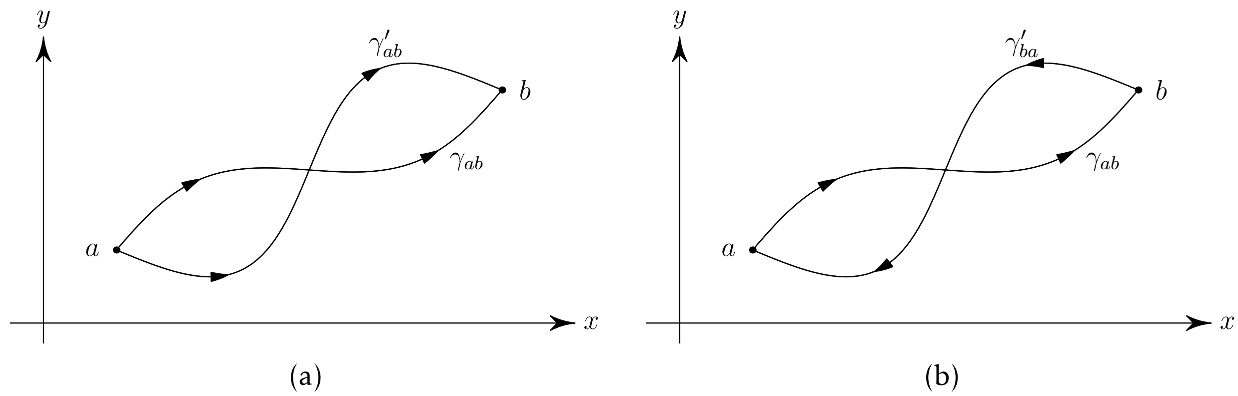

Figure 2: Two different paths, γ_{ab} and γ'_{ab}, between the points a and b (a) form a closed contour $\gamma_{ab} + \gamma'_{ba}$ (b).

where the second integral on the left can be interpreted as integrating backward on γ',

$$-\int_{\gamma'_{ab}} f(z)\,\mathrm{d}z = \int_{\gamma'_{ba}} f(z)\,\mathrm{d}z, \tag{10}$$

thus making the combined contour $\gamma_{ab} + \gamma'_{ba}$ closed. Because $f(z)$ is analytic inside this contour, the zero result follows from Cauchy theorem.

References

[1] Philip McCord Morse and Herman Feshbach. *Methods of theoretical physics, Part I*. Feshbach Publishing, 1953.

[2] A. O. Gogolin. *Lectures on Complex Integration*. Ed. by Elena G. Tsitsishvili and Andreas Komnik. Undergraduate Lecture Notes in Physics. Springer, 2014.