CAUCHY'S INTEGRAL THEOREM: EXAMPLES

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https://www.phys.uconn.edu/~rozman/Courses/P2400_25S/

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Cauchy's theorem states that if $f(z)$ is analytic at all points on and inside a closed complex contour C, then the integral of the function around that contour vanishes:

$$\oint_C f(z)\,\mathrm{d}z = 0. \tag{1}$$

1 Euler's integral

Problem: evaluate the following integral:

$$I(\alpha) = \int_0^\infty \frac{\sin(x)}{x^\alpha}\,\mathrm{d}x, \quad 0 < \alpha < 1. \tag{2}$$

Solution:

Let's consider the following integral:

$$J(\alpha) = \oint_C \frac{e^{iz}}{z^\alpha}\,\mathrm{d}z, \tag{3}$$

where the integration contour C is sketch in Fig. 1.

The integrand in Eq. (3) is an analytic function inside C, therefore

$$J(\alpha) = 0. \tag{4}$$

On the other hand,

$$J(\alpha) = J_{\mathrm{I}} + J_{\mathrm{II}} + J_{\mathrm{III}} + J_{\mathrm{IV}}. \tag{5}$$

where the subscripts corresponds to integration contours labeled in Fig. 1.

Let's consider J_{I}, J_{II}, J_{III}, and J_{IV} separately:

J_{I}**:** the integration is along the real axis, so $z = x$, $\mathrm{d}z = \mathrm{d}x$, $r \le x \le R$:

$$J_{\mathrm{I}} = \lim_{r\to 0} \lim_{R\to\infty} \int_r^R \frac{e^{ix}}{x^\alpha}\,\mathrm{d}x = \int_0^\infty \frac{e^{ix}}{x^\alpha}\,\mathrm{d}x, \tag{6}$$

so

$$I(\alpha) = \operatorname{Im} J_{\mathrm{I}}. \tag{7}$$

J_{II}**:** the integration is counterclockwise along the quarter-circle of radius R, $z = R\,e^{i\theta}$, $\mathrm{d}z = i\,R\,e^{i\theta}\mathrm{d}\theta$, $0 \le \theta \le \frac{\pi}{2}$:

$$J_{\mathrm{II}} = \lim_{R\to\infty} i\,R^{(1-\alpha)} \int_0^{\frac{\pi}{2}} e^{i\,R\cos\theta} e^{-R\sin\theta} e^{i(1-\alpha)\theta}\,\mathrm{d}\theta. \tag{8}$$

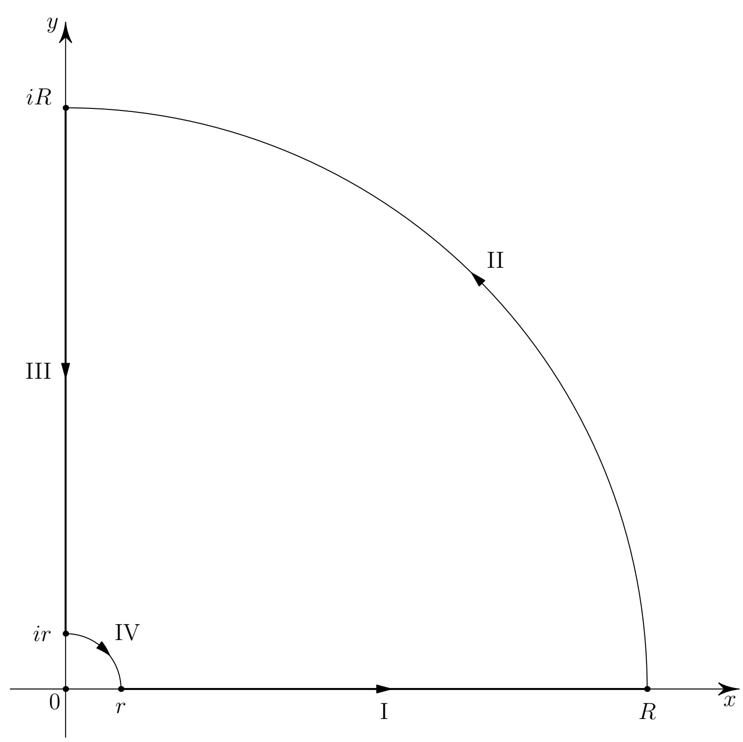

Figure 1: Integration contour for Problem 1

For the absolute value of J_{II} we have the following estimates:

$$\left|J_{\mathrm{II}}\right| = \lim_{R\to\infty}\left|R^{(1-\alpha)}\int_0^{\frac{\pi}{2}} e^{iR\cos\theta}e^{-R\sin\theta}e^{i(1-\alpha)\theta}\,\mathrm{d}\theta\right| \tag{9}$$

$$\leq \lim_{R\to\infty} R^{(1-\alpha)}\int_0^{\frac{\pi}{2}} \left|e^{iR\cos\theta}e^{-R\sin\theta}e^{i(1-\alpha)\theta}\right|\mathrm{d}\theta \tag{10}$$

$$= \lim_{R\to\infty} R^{(1-\alpha)}\int_0^{\frac{\pi}{2}} e^{-R\sin(\theta)}\,\mathrm{d}\theta \leq \lim_{R\to\infty} R^{(1-\alpha)}\int_0^{\frac{\pi}{2}} e^{-\frac{2R}{\pi}\theta}\,\mathrm{d}\theta \tag{11}$$

$$= \lim_{R\to\infty} R^{(1-\alpha)}\frac{\pi}{2R}\int_0^{R} e^{-u}\,\mathrm{d}u = \frac{\pi}{2}\lim_{R\to\infty} R^{-\alpha}\left(1-e^{-R}\right) = 0, \tag{12}$$

where we used the inequalities

$$\sin(\theta) \geq \frac{2}{\pi}\theta \quad\longrightarrow\quad e^{-\sin(\theta)} \leq e^{-\frac{2}{\pi}\theta} \quad\longrightarrow\quad e^{-R\sin(\theta)} \leq e^{-\frac{2R}{\pi}\theta}, \tag{13}$$

that are valid within the integration range $0 \leq \theta \leq \frac{\pi}{2}$, and introduce a new integration variable $u = \frac{2R}{\pi}\theta$.

Thus,

$$J_{\mathrm{II}} = 0. \tag{14}$$

J_{III}**:** the integration is along the imaginary axis, so $z = iy$, $\mathrm{d}z = i\,\mathrm{d}y$, $r \leq y \leq R$:

$$J_{\mathrm{I}} = \lim_{r\to 0}\lim_{R\to\infty} i^{(1-\alpha)}\int_R^{r}\frac{e^{-y}}{y^{\alpha}}\,\mathrm{d}y = -e^{i\frac{\pi}{2}(1-\alpha)}\int_0^{\infty} e^{-y}\,y^{-\alpha}\,\mathrm{d}y = -e^{i\frac{\pi}{2}(1-\alpha)}\,\Gamma(1-\alpha). \tag{15}$$

J_{IV}**:** the integration is clockwise along the quarter-circle of radius r, $z = r\,e^{i\theta}$, $\mathrm{d}z = i\,r\,e^{i\theta}\mathrm{d}\theta$, $0 \leq \theta \leq \frac{\pi}{2}$:

$$J_{\mathrm{IV}} = \lim_{r\to 0}\, i\, r^{(1-\alpha)}\int_{\frac{\pi}{2}}^{0} e^{i\,r\,e^{i\theta}}\,e^{i(1-\alpha)\theta}\mathrm{d}\theta \approx -\lim_{r\to 0}\, i\, r^{(1-\alpha)}\int_0^{\frac{\pi}{2}} e^{i(1-\alpha)\theta}\mathrm{d}\theta = 0. \tag{16}$$

Combining Eqs. (4), (5), and (16), we get

$$J_{\rm I} = e^{i\frac{\pi}{2}(1-\alpha)}\Gamma(1-\alpha). \tag{17}$$

Taking the imaginary part, and using Eq. (7), we obtain

$$\int_0^\infty \frac{\sin(x)}{x^\alpha}\,dx = \sin\left(\frac{\pi}{2}(1-\alpha)\right)\Gamma(1-\alpha). \tag{18}$$

For the case $\alpha = 1$,

$$\begin{aligned}\int_0^\infty \frac{\sin(x)}{x}\,dx &= \lim_{\alpha\to 1}\sin\left(\frac{\pi}{2}(1-\alpha)\right)\Gamma(1-\alpha) = \lim_{\alpha\to 1}\frac{\pi}{2}(1-\alpha)\Gamma(1-\alpha) \\ &= \frac{\pi}{2}\lim_{\alpha\to 1}\Gamma(2-\alpha) = \frac{\pi}{2}\Gamma(1) = \frac{\pi}{2}.\end{aligned} \tag{19}$$

Integral Eq. (19) is known as the *Dirichlet integral*, after the German mathematician P. Dirichlet.

For the case $\alpha = 0$, Eq. (18) gives

$$\int_0^\infty \sin(x)\,dx = \sin\left(\frac{\pi}{2}\right)\Gamma(1) = 1. \tag{20}$$

The result Eq. (20) can be understood as follows:

$$\begin{aligned}\int_0^\infty \sin(x)\,dx &= \lim_{\gamma\to 0}\int_0^\infty e^{-\gamma x}\sin(x)\,dx = \lim_{\gamma\to 0}\mathrm{Im}\left[\int_0^\infty e^{-(\gamma-i)x}\,dx\right] \\ &= \lim_{\gamma\to 0}\mathrm{Im}\left[\frac{1}{\gamma-i}\right] = \lim_{\gamma\to 0}\frac{1}{\gamma^2+1} = 1.\end{aligned} \tag{21}$$

2 Fresnel integrals

Problem: Assuming that the value of the Gaussian integral is known,

$$I = \int_0^\infty e^{-x^2}\,dx = \frac{\sqrt{\pi}}{2}, \tag{22}$$

evaluate the *Fresnel integrals*,

$$C = \int_0^\infty \cos\left(x^2\right) \mathrm{d}x \tag{23}$$

and

$$S = \int_0^\infty \sin\left(x^2\right) \mathrm{d}x. \tag{24}$$

The integrals C and S are named after the Fresnel (French physicist, 1788-1827). They were first evaluated by Euler in 1781.

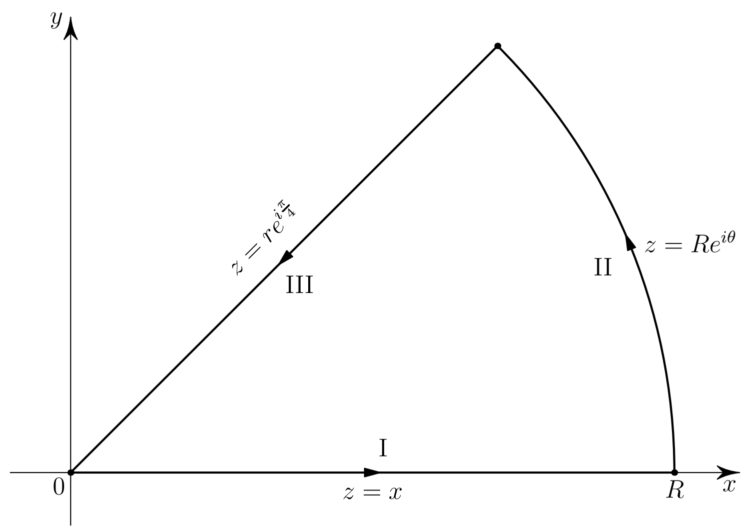

Figure 2: Integration contour for Problem 2

Solution:

Let's pack C and S together:

$$F \equiv C + i\,S = \int_0^\infty \left[\cos\left(x^2\right) + i\sin\left(x^2\right)\right] \mathrm{d}x = \int_0^\infty e^{ix^2}\,\mathrm{d}x, \tag{25}$$

such that

$$C = \operatorname{Re} F \tag{26}$$

and

$$S = \operatorname{Im} F. \tag{27}$$

Consider the integral

$$J = \int_C e^{i\,z^2}\,\mathrm{d}z, \tag{28}$$

where C is the contour in the complex plane shown in Fig. 2.

Since the integrand in Eq. (28) is analytic inside C,

$$J = 0. \tag{29}$$

On the other hand,

$$J = J_\mathrm{I} + J_\mathrm{II} + J_\mathrm{III}, \tag{30}$$

where J_I is the integral along the positive real axis, J_II is the integral along the circular arc or radius $R \to \infty$, $0 \le \theta \le \frac{\pi}{4}$, and J_III is the integral from infinity to the origin along the ray that makes the angle $\theta = \frac{\pi}{4}$ with the real axis.

Let's consider J_I, J_II, and J_III separately:

J_I**:** the integration is along the real axis, so $z = x$, $\mathrm{d}z = \mathrm{d}x$, $0 \le x \le \infty$:

$$J_\mathrm{I} = \int_{C_\mathrm{I}} e^{i\,z^2}\,\mathrm{d}z = \int_0^\infty e^{i\,x^2}\,\mathrm{d}x = F. \tag{31}$$

J_{II}**:** the integration is along the circular arc of radius R so $z = R\,e^{i\theta}$, $\mathrm{d}z = i\,R\,e^{i\theta}\mathrm{d}\theta$, $z^2 = R^2\,e^{2i\theta} = R^2\left(\cos(2\theta) + i\sin(2\theta)\right)$, $0 \le \theta \le \frac{\pi}{4}$:

$$J_\mathrm{II} = \int_{C_\mathrm{II}} e^{i\,z^2}\,\mathrm{d}z = i\,R\int_0^{\frac{\pi}{4}} e^{i\,R^2\cos(2\theta)}e^{-R^2\sin(2\theta)}\,\mathrm{d}\theta. \tag{32}$$

For the absolute value of $J_{\rm II}$ we have the following estimates:

$$\left|J_{\rm II}\right| = \left| R\int_0^{\frac{\pi}{4}} e^{i\,R^2\cos(2\theta)} e^{-R^2\sin(2\theta)}\,\mathrm{d}\theta \right| \le R\int_0^{\frac{\pi}{4}} \left| e^{i\,R^2\cos(2\theta)} e^{-R^2\sin(2\theta)} \right| \mathrm{d}\theta \qquad (33)$$

$$= R\int_0^{\frac{\pi}{4}} e^{-R^2\sin(2\theta)}\,\mathrm{d}\theta = \frac{R}{2}\int_0^{\frac{\pi}{2}} e^{-R^2\sin(\phi)}\,\mathrm{d}\phi < \frac{R}{2}\int_0^{\frac{\pi}{2}} e^{-\frac{2R^2}{\pi}\phi}\,\mathrm{d}\phi \qquad (34)$$

$$= \frac{R}{2}\,\frac{\pi}{2R^2}\int_0^{R^2} e^{-u}\,\mathrm{d}u = \frac{\pi}{4R}\left(1 - e^{-R^2}\right) < \frac{\pi}{4R}, \qquad (35)$$

where we introduced a new integration variable $\phi = 2\theta$, used the inequalities

$$\sin(\phi) \ge \frac{2}{\pi}\phi \quad \longrightarrow \quad e^{-\sin(\phi)} \le e^{-\frac{2}{\pi}\phi} \quad \longrightarrow \quad e^{-R^2\sin(\phi)} \le e^{-\frac{2R^2}{\pi}\phi}, \qquad (36)$$

that are valid within the integration range $0 \le \phi \le \frac{\pi}{2}$, and introduce a new integration variable $u = \frac{2R^2}{\pi}\phi$.

Thus we obtained that

$$\left|J_{\rm II}\right| < \frac{\pi}{4R}. \qquad (37)$$

Therefore,

$$J_{\rm II} = 0 \qquad (38)$$

as $R \to \infty$.

$J_{\rm III}$: the integration is along the ray making the angle $\frac{\pi}{4}$ with the real axis so $z = re^{i\frac{\pi}{4}}$, $z^2 = r^2 e^{i\frac{\pi}{2}} = i\,r^2$, $\mathrm{d}z = e^{i\frac{\pi}{4}}\,\mathrm{d}r$, $0 \le r < \infty$.

$$J_{\rm III} = \int_{C_{\rm III}} e^{i\,z^2}\,\mathrm{d}z = e^{i\frac{\pi}{4}}\int_{\infty}^{0} e^{-r^2}\,\mathrm{d}r = -e^{i\frac{\pi}{4}}\int_0^{\infty} e^{-r^2}\,\mathrm{d}r = -e^{i\frac{\pi}{4}}\,\frac{\sqrt{\pi}}{2}. \qquad (39)$$

Combining Eqs. (29), (31), (38), and (39) we obtain:

$$F = \frac{\sqrt{\pi}}{2}\,e^{i\frac{\pi}{4}}. \qquad (40)$$

Finally, the Fresnel's integrals are:

$$C = \operatorname{Re} F = \frac{\sqrt{\pi}}{2} \cos\left(\frac{\pi}{4}\right) = \sqrt{\frac{\pi}{8}} \tag{41}$$

and

$$S = -\operatorname{Im} F = \frac{\sqrt{\pi}}{2} \sin\left(\frac{\pi}{4}\right) = \sqrt{\frac{\pi}{8}}. \tag{42}$$