NONLINEAR OSCILLATORS. METHOD OF AVERAGING

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https://www.phys.uconn.edu/~rozman/Courses/P2400_25S/

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1 Oscillator with nonlinear friction

Let's consider the following second order non-linear differential equation

$$\frac{\mathrm{d}^2x}{\mathrm{d}t^2} + \varepsilon\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^3 + x = 0, \quad \varepsilon > 0 \tag{1}$$

with the initial conditions

$$x(0) = x_0\,, \qquad \dot{x}(0) = v_0\,. \tag{2}$$

Here and below we interchangeably use the notations $\dot{x}$ and $\frac{\mathrm{d}x}{\mathrm{d}t}$ for the first derivative with respect to time-like independent variable t, and $\ddot{x}$ and $\frac{\mathrm{d}^2x}{\mathrm{d}t^2}$ for the second derivative.

The equation describes a non-liner oscillator with the "friction" force that is proportional to the third power of the velocity. The parameter ε is a positive parameter that describes the rate of the energy loss, $\mathrm{d}E/\mathrm{d}t$, in the system ($\mathrm{d}E/\mathrm{d}t \sim \varepsilon$, see Eq. (12)). Equation (1) has no exact analytic solutions, therefore below we compare our analytics with the results of numerical calculations.

1.1 Numerical integration

To solve Eq. (1) numerically, we introduce a new dependent variable, $y = \dot{x}$ and rewrite Eq. (1) as a system of two first order differential equations for two unknown $x(t)$ and $y(t)$,

$$\begin{cases} \dfrac{\mathrm{d}x}{\mathrm{d}t} &= y, \\ \dfrac{\mathrm{d}y}{\mathrm{d}t} &= -\varepsilon y^3 - x. \end{cases} \tag{3}$$

A typical result of the numerical integration of Eqs. (3) is presented in Fig. 1.

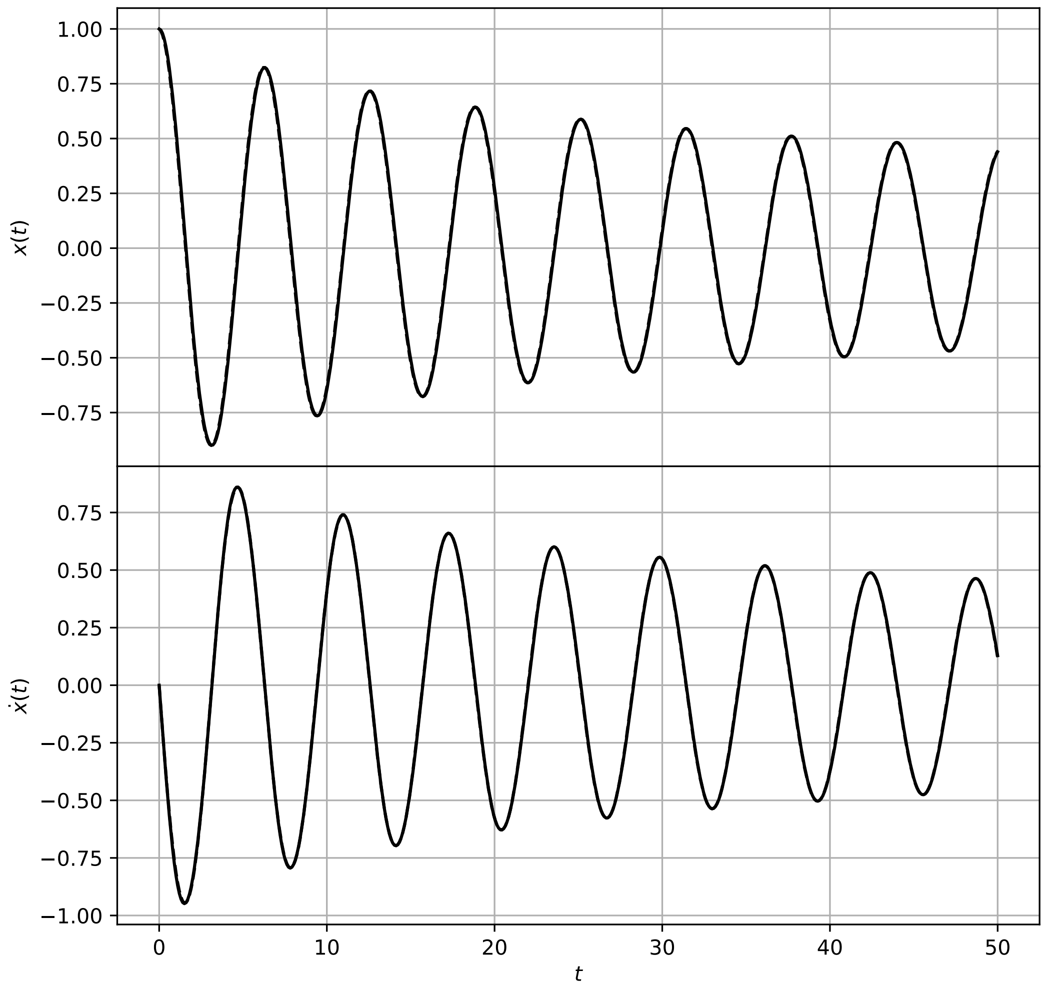

Figure 1: Typical solution of Eq. (1) for weak nonlinearity: $\varepsilon = 0.2$ (solid line). The initial conditions are $x(0) = 1$, $\dot{x}(0) = 0$. The approximation Eq. (37) is also shown (dashed line).

1.2 Regular perturbation theory for nonlinear oscillator

$$\ddot{x} + x = -\varepsilon \dot{x}^3. \tag{4}$$

A perturbative solution of this equation is obtained by expanding $x(t)$ as a power series in ε:

$$x = x_0 + \varepsilon x_1 + \varepsilon^2 x_2 + \ldots, \tag{5}$$

where $x_0(0)=1$, $\dot{x}_0(0)=0$, and $x_n(0)=0$, $\dot{x}_n(0)=0$ for $n \geq 1$. Substituting Eq. (5) into Eq. (4) and equating coefficients of like powers of ε gives a sequence of linear differential equations of which all but the first are inhomogeneous:

$$\begin{aligned} \ddot{x}_0 + x_0 &= 0, && (6)\\ \ddot{x}_1 + x_1 &= -\dot{x}_0^3, && (7)\\ \cdots & \quad \cdots \\ \ddot{x}_n + x_n &= -\dot{x}_{n-1}^3, && (8)\\ \cdots & \quad \cdots \end{aligned}$$

The solution of Eq. (6) which satisfies $x_0(0)=1$, $\dot{x}_0(0)=0$ is

$$x_0(t) = \cos(t). \tag{9}$$

The differential equation Eq. (7) for the first correction, x_1, is then as follows:

$$\ddot{x}_1 + x_1 = -\dot{x}_0^3 = \sin^3(t) = \frac{3}{4}\sin(t) - \frac{1}{4}\sin(3t), \qquad x_1(0)=0, \quad \dot{x}_1(0)=0. \tag{10}$$

Its solution,

$$x_1(t) = \frac{9}{32}\sin(t) + \frac{1}{32}\sin(3t) - \frac{3}{8}t\cos(t). \tag{11}$$

There is a serious problem with the solution Eq. (11). The amplitude of oscillation of the solution for $x_1(t)$ grows unbounded as $t \to \infty$. The term $t\cos(t)$ in the solution, whose absolute value grows with t, is said to be a *secular term*. The secular term has appeared because $\sin^3(t)$ on the right of Eq. (10) contains a component, $\sim \sin(t)$, whose frequency equals the natural frequency of the unperturbed oscillator, i.e. because the inhomogeneous term $\sim \sin(t)$ is itself a solution of the homogeneous equation associated with Eq. (10): $\ddot{x}_1 + x_1 = 0$. In general, secular terms always appear whenever the inhomogeneous term is itself a solution of the associated homogeneous constant-coefficient differential equation. A secular term always grows more rapidly than the corresponding solution of the homogeneous equation by at least a factor of t.

However, the correct solution of Eq. (4), $x(t)$, remains bounded for all t. Indeed, let's multiply Eq. (4) by $\dot{x}$.

$$\dot{x}\ddot{x} + \dot{x}x = -\varepsilon\dot{x}^4. \tag{12}$$

Rearraging terms in the left hand side, we obtain:

$$\dot{x}\ddot{x} + \dot{x}x = \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2 + \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}x^2 = \frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{1}{2}\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2 + \frac{1}{2}x^2\right] = \frac{\mathrm{d}E}{\mathrm{d}t}, \tag{13}$$

where

$$E \equiv \frac{1}{2}\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2 + \frac{1}{2}x^2 = \frac{1}{2}\dot{x}^2 + \frac{1}{2}x^2 \tag{14}$$

is the *mechanical energy* of the oscillator. The energy Eq. (14) is always non-negative.

From Eqs. (12), (13),

$$\frac{\mathrm{d}E}{\mathrm{d}t} = -\varepsilon\dot{x}^4 \leq 0. \tag{15}$$

Therefore,

$$E(t) \leq E(0) \tag{16}$$

which means that that neither $x(t)$ nor $\dot{x}$ can grow unbounded, in contradiction with the result Eq. (11).

1.3 The method of averaging

To obtain an approximate analytic solution of Eq. (1), instead of perturbation theory, we use a powerful method called the *method of averaging*. It is applicable to equations of the following general form:

$$\frac{\mathrm{d}^2x}{\mathrm{d}t^2} + \varepsilon F\left(x, \frac{\mathrm{d}x}{\mathrm{d}t}\right) + x = 0, \tag{17}$$

where in our case

$$F\left(x, \frac{\mathrm{d}x}{\mathrm{d}t}\right) = \left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^3. \tag{18}$$

We seek a solution to Eq. (17) in the form:

$$x = a(t)\cos\left(t + \psi(t)\right), \tag{19}$$

$$\frac{\mathrm{d}x}{\mathrm{d}t} = -a(t)\sin\left(t + \psi(t)\right). \tag{20}$$

The motivation for this ansatz is that when ε is zero, Eq. (17) has its solution of the form Eq. (19) with a and ψ constants. For small values of ε we expect the same form of the solution to be approximately valid, but now a and ψ are expected to be slowly varying functions of t.

Differentiating Eq. (19) and requiring Eq. (20) to hold, we obtain the following relation:

$$\frac{\mathrm{d}a}{\mathrm{d}t}\cos\left(t + \psi(t)\right) - a\frac{\mathrm{d}\psi}{\mathrm{d}t}\sin\left(t + \psi(t)\right) = 0. \tag{21}$$

Differentiation of Eq. (20) and substitution the result into Eq. (17) gives

$$-\frac{\mathrm{d}a}{\mathrm{d}t}\sin(t+\psi) - a\frac{\mathrm{d}\psi}{\mathrm{d}t}\cos(t+\psi) = -\varepsilon F\Big(a\cos(t+\psi), -a\sin(t+\psi)\Big), \tag{22}$$

where in our case

$$F\Big(a\cos(t+\psi), -a\sin(t+\psi)\Big) = -a^3\sin^3(t+\psi). \tag{23}$$

Solving Eqs. (21) and (22) for $\frac{\mathrm{d}a}{\mathrm{d}t}$ and $\frac{\mathrm{d}\psi}{\mathrm{d}t}$, we obtain the following system of two differential equations:

$$\frac{\mathrm{d}a}{\mathrm{d}t} = \varepsilon F\Big(a\cos(t+\psi), -a\sin(t+\psi)\Big)\sin(t+\psi) \tag{24}$$

$$\frac{\mathrm{d}\psi}{\mathrm{d}t} = \frac{\varepsilon}{a} F\Big(a\cos(t+\psi), -a\sin\Big(t+\psi\Big)\cos(t+\psi)\Big), \tag{25}$$

or, specifically to our case,

$$\frac{\mathrm{d}a}{\mathrm{d}t} = -\varepsilon a^3\sin^4(t+\psi) \tag{26}$$

$$\frac{\mathrm{d}\psi}{\mathrm{d}t} = -\varepsilon a^2\sin^3(t+\psi)\cos(t+\psi). \tag{27}$$

So far our treatment has been exact.

Now we introduce the following approximation: since ε is small, $\frac{\mathrm{d}a}{\mathrm{d}t}$ and $\frac{\mathrm{d}\psi}{\mathrm{d}t}$ are also small. Hence $a(t)$ and $\psi(t)$ are slowly varying functions of t. Thus over one cycle of oscillations the quantities $a(t)$ and $\psi(t)$ on the right hand sides of Eqs. (26) and (27) can be treated as nearly constant, and thus these right hand sides may be replaced by their averages:

$$\frac{1}{2\pi}\int_0^{2\pi}\mathrm{d}\phi\ldots, \tag{28}$$

where $\phi = t+\psi$.

Eqs. (26) and (27) become

$$\frac{\mathrm{d}a}{\mathrm{d}t} = -\varepsilon a^3\frac{1}{2\pi}\int_0^{2\pi}\mathrm{d}\phi\,\sin^4(\phi), \tag{29}$$

$$\frac{\mathrm{d}\psi}{\mathrm{d}t} = -\varepsilon a^2\frac{1}{2\pi}\int_0^{2\pi}\mathrm{d}\phi\,\sin^3(\phi)\cos(\phi). \tag{30}$$

The right hand side of Eq. (30) is zero:

$$\int_0^{2\pi} \mathrm{d}\phi\, \sin^3(\phi)\cos(\phi) = \int_0^{2\pi} \sin^3(\phi)\, \mathrm{d}(\sin(\phi)) = \frac{1}{4}\sin^4(\phi)\Big|_0^{2\pi} = 0. \tag{31}$$

The averaging in Eq. (29) can be done using the following trigonometric identity:

$$\sin^2(\phi) = \frac{1}{2}(1-\cos(2\phi)).$$

$$\frac{1}{2\pi}\int_0^{2\pi} \mathrm{d}\phi \cos^2(n\phi) = \frac{1}{2\pi}\int_0^{2\pi} \mathrm{d}\phi \sin^2(n\phi) = \frac{1}{2}, \quad n = 1, 2, \ldots$$

$$\frac{1}{2\pi}\int_0^{2\pi} \mathrm{d}\phi \cos(n\phi) = \frac{1}{2\pi}\int_0^{2\pi} \mathrm{d}\phi \sin(n\phi) = 0, \quad n = 1, 2, \ldots$$

$$\begin{aligned}\frac{1}{2\pi}\int_0^{2\pi} \mathrm{d}\phi \sin^4(\phi) &= \frac{1}{2\pi}\int_0^{2\pi} \mathrm{d}\phi \left(\frac{1}{2}(1-\cos(2\phi))\right)^2 = \\ &= \frac{1}{4}\frac{1}{2\pi}\int_0^{2\pi} \mathrm{d}\phi \left(1 - 2\cos(2\phi) + \cos^2(2\phi)\right) = \\ &= \frac{1}{4}\left(1+\frac{1}{2}\right) = \frac{3}{8}\end{aligned} \tag{32}$$

The averaged equations are as following:

$$\frac{\mathrm{d}a}{\mathrm{d}t} = -\varepsilon\frac{3}{8}a^3 \tag{33}$$

$$\frac{\mathrm{d}\psi}{\mathrm{d}t} = 0 \tag{34}$$

The solution of Eq. (34) is

$$\psi = \psi_0 = \mathrm{const}, \tag{35}$$

where ψ_0 is determined by the initial conditions.

Eq. (33) can be solved by separating the variables:

$$\frac{\mathrm{d}a}{a^3} = -\frac{3}{8}\varepsilon \mathrm{d}t \quad \longrightarrow \quad \frac{1}{a^2(t)} = \frac{3}{4}\varepsilon t + \frac{1}{a_0^2} \quad \longrightarrow \quad a(t) = \frac{1}{\sqrt{\frac{3}{4}\varepsilon t + \frac{1}{a_0^2}}}, \tag{36}$$

where $a_0 = a(0)$ is the amplitude of oscillations at $t = 0$. Finally,

$$x(t) = \frac{\cos(t+\psi_0)}{\sqrt{\frac{3}{4}\varepsilon t + \frac{1}{a_0^2}}}, \qquad \dot{x}(t) = -\frac{\sin(t+\psi_0)}{\sqrt{\frac{3}{4}\varepsilon t + \frac{1}{a_0^2}}}. \tag{37}$$

The integration constants a_0 and ψ_0 are determined from the initial conditions,

$$x(0) = a_0 \cos\psi_0, \qquad \dot{x}(0) = -a_0 \sin\psi_0, \tag{38}$$

$$a_0 = \sqrt{x^2(0) + \dot{x}^2(0)}, \qquad \psi_0 = -\frac{\dot{x}(0)}{x(0)}. \tag{39}$$

2 Van der Pol oscillator

The second order non-linear autonomous differential equation

$$\frac{\mathrm{d}^2 x}{\mathrm{d}t^2} + \varepsilon\left(x^2 - 1\right)\frac{\mathrm{d}x}{\mathrm{d}t} + x = 0, \quad \varepsilon > 0 \tag{40}$$

is called van der Pol equation. The parameter ε is positive and indicates the nonlinearity and the strength of the damping. The equation models a non-conservative system in which energy is added to and subtracted from the system, resulting in a periodic motion called a *limit cycle*. The sign of the "coefficient" in the damping term in Eq. (40), $\left(x^2 - 1\right)$ changes, depending whether $|x|$ is larger or smaller than one, describing the inflow and outflow of the energy.

The equation was originally proposed in the late 1920-th to describe stable oscillations in electrical circuits employing vacuum tubes.

Van der Pol oscillator is the example of a system that exibits the so called *limit cycle*. A limit cycle is an isolated closed trajectory. Isolated means that neighboring trajectories are not closed; they spiral either toward or away from the limit cycle. If all neighboring trajectories approach the limit cycle, we say the limit cycle is stable or attracting. Otherwise the limit cycle is in general unstable.

Stable limit cycles model systems, e.g. the beating of a heart, that exhibit self-sustained oscillations. These systems oscillate even in the absence of external periodic forcing. There is a standard oscillation of some preferred period, waveform, and amplitude. If the system is perturbed slightly, it returns to the standard cycle.

Limit cycles are inherently nonlinear phenomena. They can't occur in linear systerns. Of course, a linear system, such as a linear differential equation, can have closed orbits – periodic solutions, but they won't be isolated. If $x(t)$ is a periodic solution, then so is $\alpha x(t)$ for any constant $\alpha \neq 0$. Hence $x(t)$ is surrounded by a 'family' of closed orbits. Consequently, the amplitude of a linear oscillation is set entirely by its initial conditions. Any slight disturbance to the amplitude will persist forever. In contrast, limit cycle oscillations are determined by the structure of the system itself.

Limit cycles are only possible in systems with dissipation. System that conserve energy do not have isolated closed trajectories ...

2.1 Numerical integration

Let's write Eq. (40) as a system of first order ordinary differential equations,

$$\begin{cases} \dfrac{\mathrm{d}x}{\mathrm{d}t} &= y, \\ \dfrac{\mathrm{d}y}{\mathrm{d}t} &= -\varepsilon\left(x^2-1\right)\dfrac{\mathrm{d}x}{\mathrm{d}t} - x, \end{cases} \tag{41}$$

where we introduced a new dependent variable $y(t)$, $y(t) \equiv \frac{\mathrm{d}x}{\mathrm{d}t}$.

The results of numerical integration of Eqs. (41) for the initial conditions $x(0) = 1$, $\dot{x}(0) = y(0) = 0$, are presented in Figs. 2–3.

Numerical integration of Eq. (41) shows that every initial condition (except $x = 0$, $\dot{x} = 0$) approaches a unique periodic motion. The nature of this *limit cycle* is dependent on the value of ε. For small values of ε the motion is nearly harmonic.

Numerical integration shows that the limit cycle is a closed curve enclosing the origin in the x-y phase plane. From the fact that Eqs. (41) are invariant under the transformation $x \to -x$, $y \to -y$, we may conclude that the curve representing the limit cycle is point-symmetric about the origin.

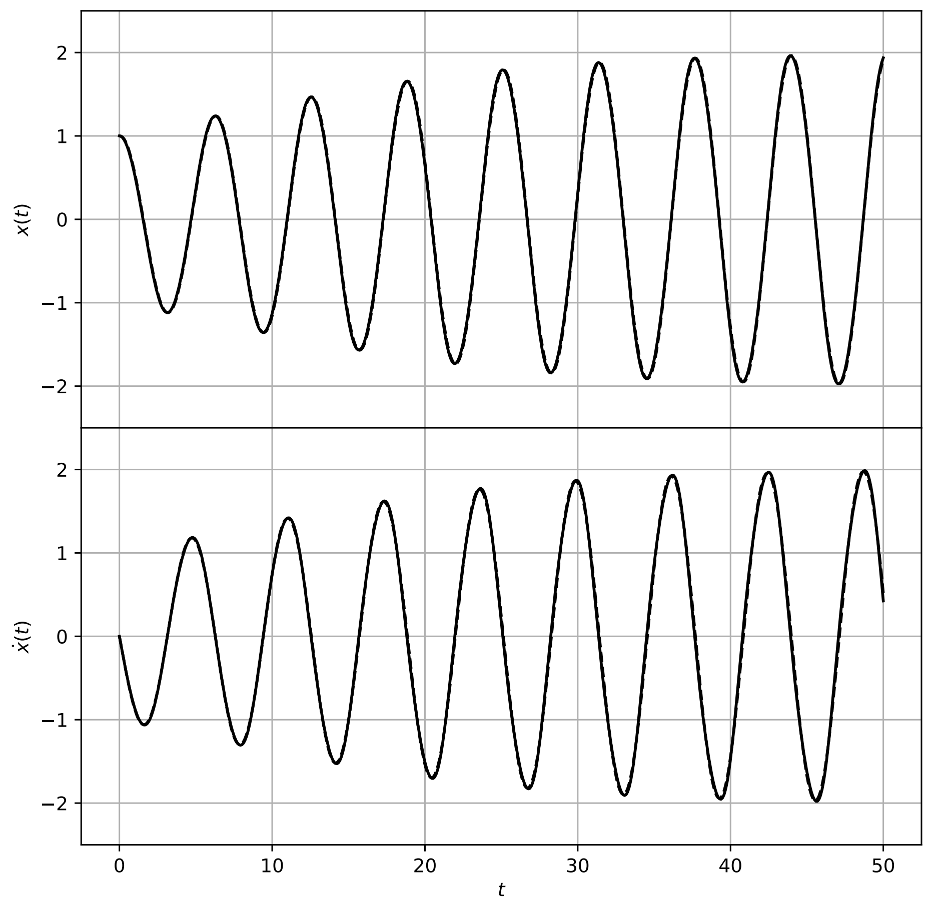

Figure 2: Typical solution of van der Pol equation for small values of ε; top graph – $x(t)$, bottom graph – $\dot{x}(t)$; $\varepsilon = 0.1$ (solid line). The approximations Eq. (75), (76) shown as dashed line.

2.2 Averaging

In order to obtain information regarding the approach to the limit cycle, we use the method of averaging. We can rewrite the van der Pol equations of the following general form:

$$\frac{\mathrm{d}^2 x}{\mathrm{d}t^2} + x = -\varepsilon F\left(x, \frac{\mathrm{d}x}{\mathrm{d}t}\right), \tag{42}$$

where

$$F\left(x, \frac{\mathrm{d}x}{\mathrm{d}t}\right) = \left(x^2 - 1\right)\frac{\mathrm{d}x}{\mathrm{d}t}. \tag{43}$$

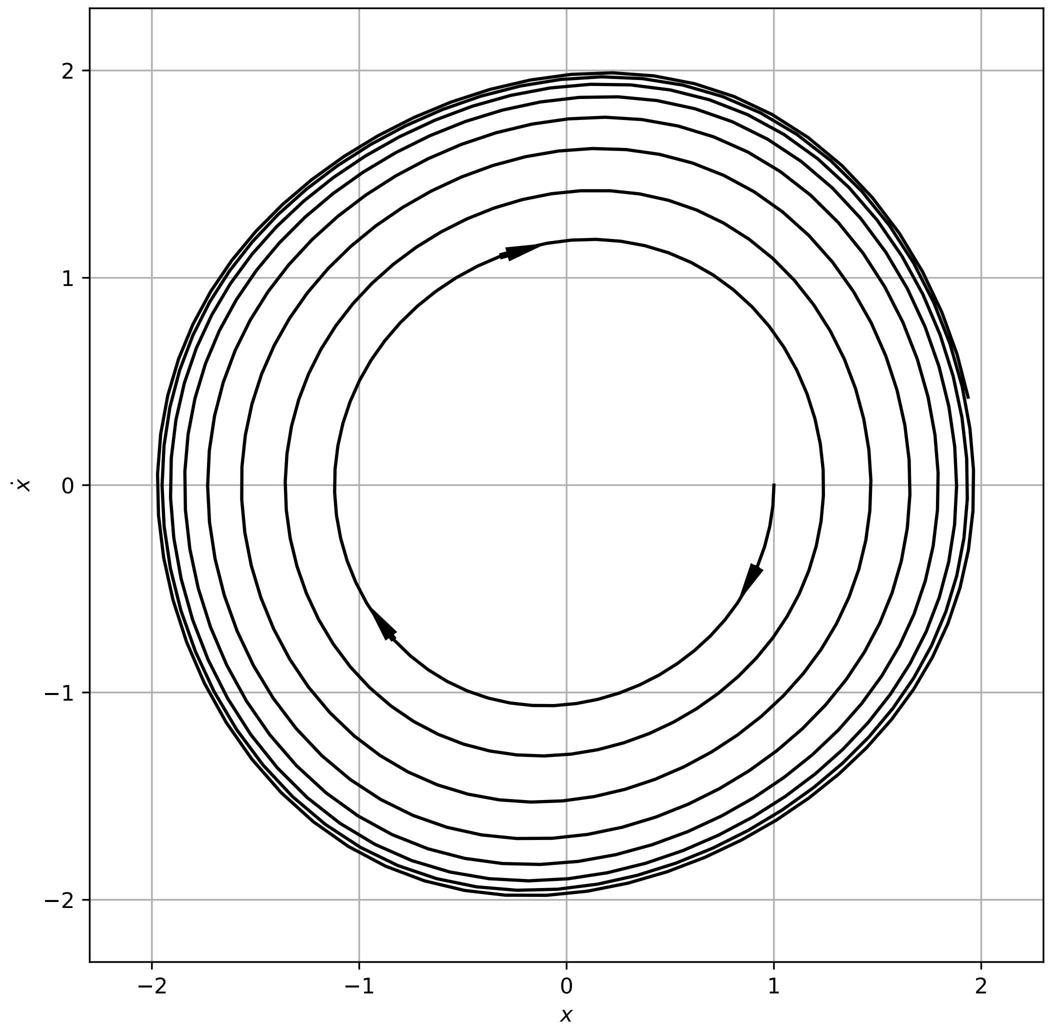

Figure 3: Typical phase space trajectory of van der Pol equation for small values of ε.

Recall that the method of averaging seeks a solution to Eq. (42) in the form:

$$x = a(t)\cos(t + \psi(t)), \tag{44}$$

$$\frac{\mathrm{d}x}{\mathrm{d}t} = -a(t)\sin(t + \psi(t)). \tag{45}$$

Our motivation for this ansatz is, as in the example before, that when ε is zero, Eq. (42) has its solution of the form Eq. (44) with a and ψ constants. For small values of ε we expect the same form of the solution to be approximately valid, but now a and ψ are expected to be slowly varying functions of t.

Differentiating Eq. (44) and requiring Eq. (45) to hold, we obtain:

$$\frac{\mathrm{d}a}{\mathrm{d}t}\cos(t+\psi(t)) - a\frac{\mathrm{d}\psi}{\mathrm{d}t}\sin(t+\psi(t)) = 0. \tag{46}$$

Differentiating Eq. (45) and substituting the result into Eq. (42) gives

$$-\frac{\mathrm{d}a}{\mathrm{d}t}\sin(t+\psi) - a\frac{\mathrm{d}\psi}{\mathrm{d}t}\cos(t+\psi) = -\varepsilon F\left(a(t)\cos(t+\psi), -a(t)\sin(t+\psi)\right). \tag{47}$$

Solving Eqs. (46) and (47) for $\frac{\mathrm{d}a}{\mathrm{d}t}$ and $\frac{\mathrm{d}\psi}{\mathrm{d}t}$, we obtain:

$$\begin{aligned}\frac{\mathrm{d}a}{\mathrm{d}t} &= \varepsilon F\left(a(t)\cos(t+\psi), -a(t)\sin(t+\psi)\right)\sin(t+\psi) &(48)\\ \frac{\mathrm{d}\psi}{\mathrm{d}t} &= \frac{\varepsilon}{a} F\left(a\cos(t+\psi), -a\sin(t+\psi)\cos(t+\psi)\right), &(49)\end{aligned}$$

where

$$F(\ldots) = -a\left(a^2\cos^2(t+\psi) - 1\right)\sin(t+\psi). \tag{50}$$

$$\begin{aligned}\frac{\mathrm{d}a}{\mathrm{d}t} &= -\varepsilon a\left(a^2\cos^2(t+\psi) - 1\right)\sin^2(t+\psi) &(51)\\ \frac{\mathrm{d}\psi}{\mathrm{d}t} &= -\varepsilon\left(a^2\cos^2(t+\psi) - 1\right)\sin(t+\psi)\cos(t+\psi) &(52)\end{aligned}$$

So far our treatment has been exact.

Now we introduce the following approximation: since ε is small, $\frac{\mathrm{d}a}{\mathrm{d}t}$ and $\frac{\mathrm{d}\psi}{\mathrm{d}t}$ are also small. Hence $a(t)$ and $\psi(t)$ are slowly varying functions of t. Thus over one cycle of oscillations the quantities $a(t)$ and $\psi(t)$ on the right hand sides of Eqs. (51) and (52) can be treated as nearly constant, and thus these right hand sides may be replaced by their averages:

$$\overline{\ldots} \equiv \langle \ldots \rangle \equiv \frac{1}{2\pi}\int_0^{2\pi} \ldots \,\mathrm{d}\phi \tag{53}$$

Eqs. (51) and (52) become

$$\begin{aligned}\frac{\mathrm{d}a}{\mathrm{d}t} &= -\varepsilon a^3\,\overline{\cos^2(\phi)\sin^2(\phi)} + \varepsilon a\,\overline{\sin^2(\phi)} &(54)\\ \frac{\mathrm{d}\psi}{\mathrm{d}t} &= -\varepsilon a^2\,\overline{\cos^3(\phi)\sin(\phi)} + \varepsilon\overline{\cos(\phi)\sin(\phi)} &(55)\end{aligned}$$

As shown in the Appendix,

$$\overline{\cos^3(\phi)\sin(\phi)} \equiv \frac{1}{2\pi} I_{3,1} = 0, \tag{56}$$

$$\overline{\cos(\phi)\sin(\phi)} \equiv \frac{1}{2\pi} I_{1,1} = 0, \tag{57}$$

thus the right hand side of Eq. (55) is zero. Therefore,

$$\frac{\mathrm{d}\psi}{\mathrm{d}t} = 0, \tag{58}$$

i.e.

$$\psi = \mathrm{const} = \psi_0, \tag{59}$$

where ψ_0 is an integration constant.

The averaged terms in Eq. (51) are as following:

$$\overline{\cos^2(\phi)\sin^2(\phi)} \equiv \frac{1}{2\pi} I_{2,2} = \frac{1}{8}, \tag{60}$$

$$\overline{\sin^2(\phi)} \equiv \frac{1}{2\pi} I_{2,0} = \frac{1}{2}, \tag{61}$$

where Eq. (123) and (124) have been used.

Thus, the averaged Eq. (54) is

$$\frac{\mathrm{d}a}{\mathrm{d}t} = \frac{\varepsilon}{8} a (4 - a^2). \tag{62}$$

Eq. (62) can be solved separating variables:

$$\frac{\mathrm{d}a}{a(2-a)(2+a)} = \frac{\varepsilon}{8}\mathrm{d}t. \tag{63}$$

Decomposing the left hand side into partial fractions,

$$\frac{1}{a(2-a)(2+a)} = \frac{1}{4}\frac{1}{a} + \frac{1}{8}\frac{1}{2-a} - \frac{1}{8}\frac{1}{2+a}, \tag{64}$$

we obtain

$$2\frac{\mathrm{d}a}{a} + \frac{\mathrm{d}a}{2-a} - \frac{\mathrm{d}a}{2+a} = \varepsilon\,\mathrm{d}t, \tag{65}$$

$$2\frac{\mathrm{d}a}{a} - \frac{\mathrm{d}(2-a)}{2-a} - \frac{\mathrm{d}(2+a)}{2+a} = \varepsilon\,\mathrm{d}t, \tag{66}$$

$$\mathrm{d}\log\left(a^2\right) - \mathrm{d}\log|2-a| - \mathrm{d}\log(2+a) = \varepsilon \mathrm{d}t. \tag{67}$$

Integrating both sides

$$\log\left(\frac{a^2}{(a+2)\,|2-a|}\right) = \varepsilon t + C_0, \tag{68}$$

where C_0 is an integration constant. Exponentiating, we obtain

$$\frac{a^2}{(a+2)\,|2-a|} = C_1 e^{\varepsilon t}, \tag{69}$$

where $C_1 = e^{C_0}$, or

$$\frac{a^2-4}{a^2} = C e^{-\varepsilon t}, \tag{70}$$

where the integration constant can now be positive or negative.

Solving Eq. (70) for $a^2(t)$, we obtain,

$$a^2(t) = \frac{4}{1 - Ce^{-\varepsilon t}}. \tag{71}$$

Letting $t = 0$ in the last equation, we obtain the relation between $a_0 = a(0)$ and the integration constant:

$$C = 1 - \frac{4}{a_0^2}. \tag{72}$$

Finally,

$$a^2(t) = \frac{4}{1 + \left(\frac{4}{a_0^2} - 1\right)e^{-\varepsilon t}} \tag{73}$$

and

$$a(t) = \frac{2}{\sqrt{1 + \left(\frac{4}{a_0^2} - 1\right)e^{-\varepsilon t}}}. \tag{74}$$

Finally,

$$x(t) = a(t)\cos(t + \psi(t)) = \frac{2\cos(t+\psi_0)}{\sqrt{1 + \left(\frac{4}{a_0^2} - 1\right)e^{-\varepsilon t}}}, \tag{75}$$

$$\dot{x}(t) = -a(t)\sin(t + \psi(t)) = -\frac{2\sin(t+\psi_0)}{\sqrt{1 + \left(\frac{4}{a_0^2} - 1\right)e^{-\varepsilon t}}}. \tag{76}$$

Since we are primary interested in the limit cycle solution of the van der Pol equation, let's consider the limit of the solution for large values of t. As $t \to \infty$ $e^{-\varepsilon t} \to 0$ therefore

$$a(t) \approx 2 - \left(\frac{4}{a_0^2} - 1\right) e^{-\varepsilon t}. \tag{77}$$

This, for large t, $t \geq \varepsilon^{-1}$,

$$x(t) = a(t)\cos(t + \psi(t)) = \left(2 - \left(\frac{4}{a_0^2} - 1\right)\right)\cos(t + \psi_0), \tag{78}$$

$$\dot{x}(t) = -a(t)\sin(t + \psi(t)) = -\left(2 - \left(\frac{4}{a_0^2} - 1\right)\right)\sin(t + \psi_0), \tag{79}$$

are the parametric equations of the limit cycle in the phase plane.

3 Oscillator with the slowly changing frequency

The technique of averaging is applicable to nonlinear oscillators that are described by differential equations with slow changing explicit time-dependent terms:

$$\frac{\mathrm{d}^2 x}{\mathrm{d}t^2} + x = -\varepsilon F\left(x, \frac{\mathrm{d}x}{\mathrm{d}t}, \boldsymbol{\varepsilon t}\right), \tag{80}$$

Here the new slow time dependence in the non-linear term is highlighted in bold.

Consider the oscillator with the slowly changing frequency.

$$\frac{\mathrm{d}^2 x}{\mathrm{d}t^2} + \omega^2(\varepsilon t)\, x = 0, \tag{81}$$

where

$$\omega(\varepsilon t) \neq 0. \tag{82}$$

To reduce Eq. (81) to the form (81), consider the change of independent variable t:

$$\tau = f(t) \tag{83}$$

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \frac{\mathrm{d}x}{\mathrm{d}\tau}\frac{\mathrm{d}\tau}{\mathrm{d}t} = \frac{\mathrm{d}f}{\mathrm{d}t}\frac{\mathrm{d}x}{\mathrm{d}\tau}, \tag{84}$$

$$\begin{aligned}\frac{d^2x}{dt^2} &= \frac{d}{dt}\left(\frac{dx}{dt}\right) = \frac{d}{dt}\left(\frac{df}{dt}\frac{dx}{d\tau}\right) = \frac{d^2f}{dt^2}\frac{dx}{d\tau} + \frac{df}{dt}\frac{d}{dt}\left(\frac{dx}{d\tau}\right) \\ &= \frac{d^2f}{dt^2}\frac{dx}{d\tau} + \frac{df}{dt}\frac{d}{d\tau}\left(\frac{dx}{d\tau}\right)\frac{df}{dt} = \frac{d^2f}{dt^2}\frac{dx}{d\tau} + \left(\frac{df}{dt}\right)^2\frac{d^2x}{d\tau^2} \qquad (85)\end{aligned}$$

Substituting Eq. (85) into Eq. (81) and introducing the notations

$$\dot{x} = \frac{dx}{d\tau}, \quad \ddot{x} = \frac{d^2x}{d\tau^2}, \qquad (86)$$

$$\left(\frac{df}{dt}\right)^2 \ddot{x} + \frac{d^2f}{dt^2}\dot{x} + \omega^2(\varepsilon t)x = 0. \qquad (87)$$

Let's choose

$$\left(\frac{df}{dt}\right)^2 = \omega^2(\varepsilon t) \quad \rightarrow \quad \frac{df}{dt} = \omega(\varepsilon t), \qquad (88)$$

then

$$\tau = \int^t \omega(\varepsilon u)\,du, \quad d\tau = \omega(\varepsilon t)\,dt \qquad (89)$$

$$\frac{d^2f}{dt^2} = \frac{d\omega(\varepsilon t)}{dt} = \varepsilon\omega'(T), \qquad (90)$$

where

$$T = \varepsilon t. \qquad (91)$$

Eq. (87) can be written as following:

$$\frac{d^2x}{d\tau^2} + \varepsilon\frac{\omega'(\varepsilon t)}{\omega^2(\varepsilon t)}\frac{dx}{d\tau} + x = 0. \qquad (92)$$

Eq. (92) is in the form Eq. (80). Using the method of averaging we obtain the following equations for $a(t)$ and $\psi(t)$:

$$\frac{da}{d\tau} = -\varepsilon\frac{\omega'(\varepsilon t)}{\omega^2(\varepsilon t)}a\sin^2(\tau+\psi), \qquad (93)$$

$$\frac{d\psi}{d\tau} = -\varepsilon\frac{\omega'(\varepsilon t)}{\omega^2(\varepsilon t)}a\sin(\tau+\psi)\cos(\tau+\psi). \qquad (94)$$

Averaging Eq. (93), (94) we obtain:

$$\frac{da}{d\tau} = -\frac{\varepsilon}{2}\frac{\omega'(\varepsilon t)}{\omega^2(\varepsilon t)}a, \qquad (95)$$

$$\frac{d\psi}{d\tau} = 0. \qquad (96)$$

Eq. (96) tells us that $\psi = \text{const}$, and we can chose

$$\psi = 0. \tag{97}$$

Eq. (95) can be solved separating variables

$$\frac{\mathrm{d}a}{a} = -\frac{\varepsilon}{2}\frac{\omega'(\varepsilon t)}{\omega^2(\varepsilon t)}\,\mathrm{d}\tau = -\frac{\varepsilon}{2}\frac{\omega'(\varepsilon t)}{\omega^2(\varepsilon t)}\,\omega(\varepsilon t)\,\mathrm{d}t = -\frac{1}{2}\frac{\omega'(\varepsilon t)}{\omega(\varepsilon t)}\,\mathrm{d}(\varepsilon t). \tag{98}$$

$$\mathrm{d}\log(a) = -\frac{1}{2}\,\mathrm{d}\log(\omega(\varepsilon t)), \tag{99}$$

$$\log(a) = \log\left(\frac{1}{\sqrt{\omega(\varepsilon t)}}\right) + C', \tag{100}$$

$$a = \frac{C}{\sqrt{\omega(\varepsilon t)}} \tag{101}$$

$$x(t) = a(t)\cos(\tau) = \frac{C}{\sqrt{\omega(\varepsilon t)}}\cos\left(\int_0^t \omega(\varepsilon t')\,\mathrm{d}t'\right). \tag{102}$$

$$\dot{x}(t) = -C\sqrt{\omega(\varepsilon t)}'\sin\left(\int_0^t \omega(\varepsilon t')\,\mathrm{d}t'\right). \tag{103}$$

$$E(t) = \frac{1}{2}\dot{x}^2 + \frac{1}{2}\omega(\varepsilon t)^2 x^2 = \frac{C^2}{2}\omega, \tag{104}$$

$$\frac{E(t)}{\omega(\varepsilon t)} = \text{const}. \tag{105}$$

4 Problems

Problem 1. Find (a) the time dependence of the amplitude and (b) the frequency of the Duffing oscillator:

$$\ddot{x} + x + \epsilon x^3 = 0, \tag{106}$$

where ϵ is a small parameter ($\epsilon \ll 1$); $x(0) = 1$, $\dot{x}(0) = 0$. Compare your analytic approximation with the numerical solution of the differential equation.

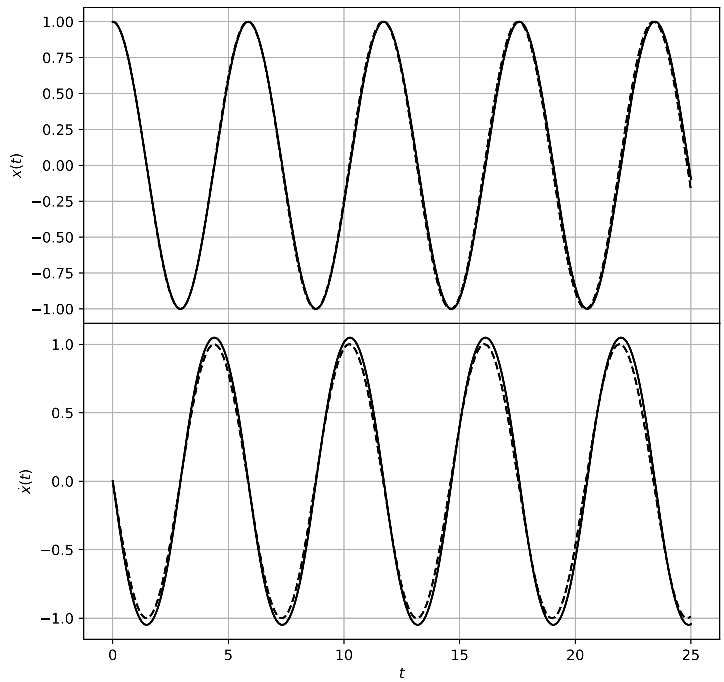

Figure 4: Typical solution of the Duffing equation Eq. (106), $\varepsilon = 0.2$ (solid line). The approximation obtained by the method of averaging is also shown (dashed line).

Problem 2. Find the time dependence of the amplitude of an oscillator with "dry" friction:

$$\ddot{x} + \gamma\,\mathrm{sign}(\dot{x}) + x = 0, \tag{107}$$

where γ is a small parameter ($\gamma \ll 1$); $x(0) = 1$, $\dot{x}(0) = 0$,

$$\mathrm{sign}(\alpha) = \begin{cases} 1, & \alpha > 0, \\ 0, & \alpha = 0, \\ -1 & \alpha < 0. \end{cases}$$

Determine the time until the full stop.

Compare your analytic approximation with the numerical solution of the differential equation.

Problem 3. Find the solution of the following nonlinear differential equation:

$$\ddot{x} + \epsilon \dot{x}^5 + x = 0, \quad x(0) = x_0, \quad \dot{x}(0) = 0, \tag{108}$$

where ϵ is a small positive parameter. Compare your analytic approximation with the numerical solution of the differential equation.

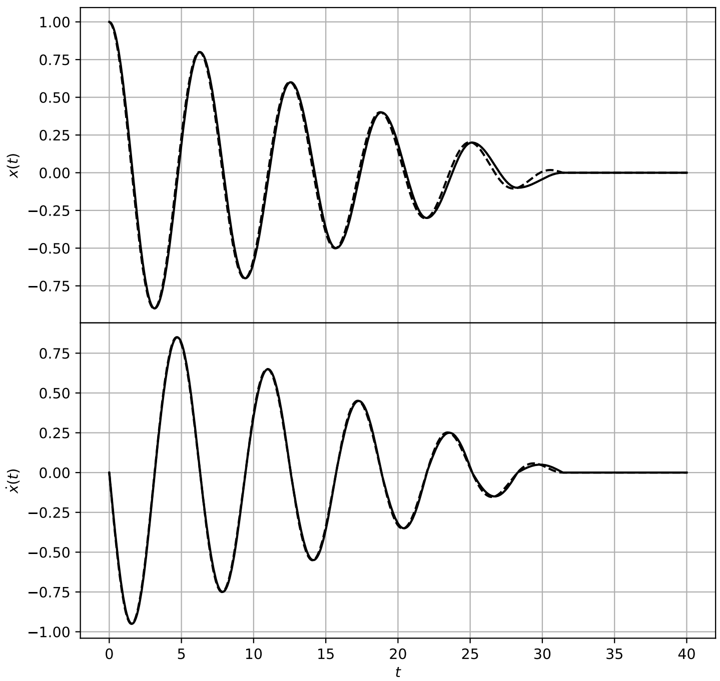

Figure 5: Typical solution of the dry friction oscillator Eq. (107) for small values of γ; $\gamma = 0.05$ (solid line). The approximation obtained by the method of averaging is also shown (dashed line).

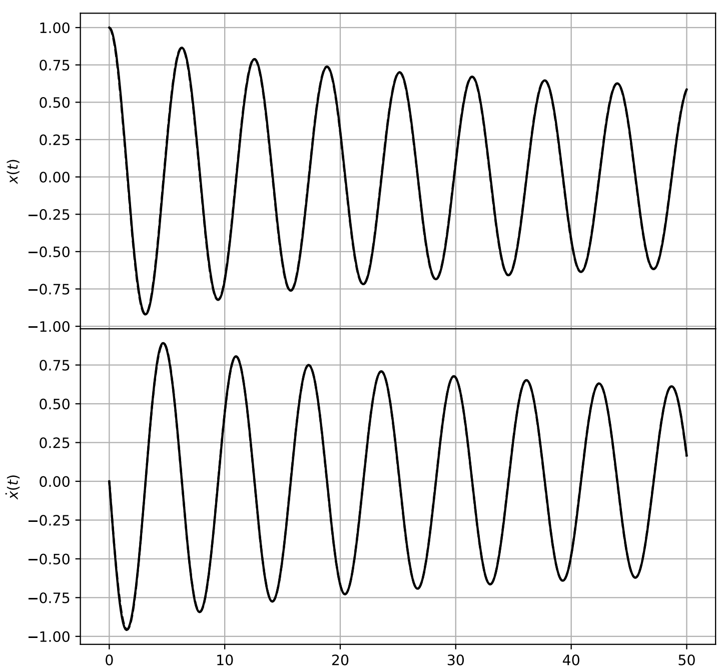

Figure 6: Typical solution of the nonlinear friction oscillator Eq. (108) for small values of ϵ; $\gamma = 0.1$ (solid line). The approximation obtained by the method of averaging is also shown (dashed line).

Appendix A. Integrals for the method of averaging

The method of averaging requires the evaluation of integrals of the form

$$I_{p,q} = \int_0^{2\pi} \cos^p x \sin^q x \, \mathrm{d}x, \tag{109}$$

where p and q are positive integers.

First, notice that the integration in Eq. (109) is over the period of the integrand, thus

$$\int_0^{2\pi} \cos^p x \sin^q x \, \mathrm{d}x = \int_u^{2\pi+u} \cos^p x \sin^q x \, \mathrm{d}x \tag{110}$$

for arbitrary u.

$I_{p,q}$ is zero if at least one of p or q is odd. Indeed, consider separately the three possible cases:

1. If p is even and q is odd, i.e. if $p = 2m$ and $q = 2n+1$, then

$$I_{2m,2n+1} = \int_0^{2\pi} \cos^{2m}(x) \sin^{2n+1}(x) \, \mathrm{d}x = \int_{-\pi}^{\pi} \cos^{2m}(x) \sin^{2n+1}(x) \, \mathrm{d}x = 0 \tag{111}$$

   since the integrand is an odd function.

2. If both p and q are odd, i.e. $p = 2m+1$ and $q = 2n+1$, then

$$I_{2m+1,2n+1} = \int_0^{2\pi} \cos^{2m+1}(x) \sin^{2n+1}(x) \, \mathrm{d}x = \frac{1}{2} \int_{-\pi}^{\pi} \cos^{2m}(x) \sin^{2n}(x) \sin(2x) \, \mathrm{d}x = 0 \tag{112}$$

   since the integrand is again an odd function; here we used the identity $\cos(x)\sin(x) = \frac{1}{2}\sin(2x)$.

3. If p is odd and q is even, i.e. if $p = 2m+1$ and $q = 2n$, then, using the identities

$\sin(x) = \cos\left(x - \frac{\pi}{2}\right)$ and $\cos(x) = -\sin\left(x - \frac{\pi}{2}\right)$,

$$\begin{aligned} I_{2m+1,2n} &= \int_0^{2\pi} \cos^{2m+1}(x)\sin^{2n}(x)\,\mathrm{d}x = -\int_0^{2\pi} \sin^{2m+1}\left(x-\frac{\pi}{2}\right)\cos^{2n}\left(x-\frac{\pi}{2}\right)\mathrm{d}x \\ &= -\int_{-\frac{1}{2}\pi}^{\frac{3}{2}\pi} \sin^{2m+1}(u)\cos^{2n}(u)\,\mathrm{d}u = -\int_{-\pi}^{\pi} \sin^{2m+1}(u)\cos^{2n}(u)\,\mathrm{d}u = 0 \end{aligned} \tag{113}$$

since the last integral is from an odd function.

To evaluate I_{pq} when both p and q are even, let's proceed as following.

$$\begin{aligned} I_{2m,2n} &= \int_0^{2\pi} \left(\cos^2(x)\right)^m \left(\sin^2(x)\right)^n \mathrm{d}x = 2\int_0^{\pi} \left(\cos^2(x)\right)^m \left(1-\cos^2(x)\right)^n \mathrm{d}x \\ &= 2\int_0^{\frac{\pi}{2}} \left(\cos^2(x)\right)^m \left(1-\cos^2(x)\right)^n \mathrm{d}x + 2\int_{\frac{\pi}{2}}^{\pi} \left(\cos^2(x)\right)^m \left(1-\cos^2(x)\right)^n \mathrm{d}x. \end{aligned} \tag{114}$$

Let's introduce the new integration variable,

$$u = \cos^2 x, \quad 0 \le u \le 1, \quad \mathrm{d}u = -2\cos x \sin x\,\mathrm{d}x. \tag{115}$$

In the first integral in Eq. (115), $0 \le x \le \frac{\pi}{2}$, thus both $\cos(x)$ and $\sin(x)$ are positive, therefore $\cos(x) = u^{\frac{1}{2}}$ and $\sin(x) = (1-u)^{\frac{1}{2}}$. So,

$$\mathrm{d}u = -2u^{\frac{1}{2}}(1-u)^{\frac{1}{2}}\,\mathrm{d}x, \tag{116}$$

i.e.

$$\mathrm{d}x = -\frac{\mathrm{d}u}{2u^{\frac{1}{2}}(1-u)^{\frac{1}{2}}}. \tag{117}$$

In the second integral in Eq. (115), $\frac{\pi}{2} \le x \le \pi$, thus $\cos(x)$ is negative and $\sin(x)$ is positive, therefore $\cos(x) = -u^{\frac{1}{2}}$ and $\sin(x) = (1-u)^{\frac{1}{2}}$. So,

$$\mathrm{d}u = 2u^{\frac{1}{2}}(1-u)^{\frac{1}{2}}\,\mathrm{d}x, \tag{118}$$

i.e.

$$\mathrm{d}x = \frac{\mathrm{d}u}{2u^{\frac{1}{2}}(1-u)^{\frac{1}{2}}}. \tag{119}$$

Substituting Eqs. (118)–(119) into Eq. (114), we obtain

$$I_{2m,2n} = -\int_1^0 u^{m-\frac{1}{2}}(1-u)^{n-\frac{1}{2}}\mathrm{d}u + \int_0^1 u^{m-\frac{1}{2}}(1-u)^{n-\frac{1}{2}}\mathrm{d}u = 2\int_0^1 u^{m+\frac{1}{2}-1}(1-u)^{n+\frac{1}{2}-1}\mathrm{d}u. \tag{120}$$

The last integral is $B\left(m+\frac{1}{2}, n+\frac{1}{2}\right)$, therefore

$$I_{2m,2n} = 2B\left(m+\frac{1}{2}, n+\frac{1}{2}\right) = \frac{2\Gamma\left(m+\frac{1}{2}\right)\Gamma\left(n+\frac{1}{2}\right)}{\Gamma(m+n+1)}. \tag{121}$$

In particular, $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$ and $\Gamma(1) = 1$, thus

$$I_{0,0} = \frac{2\Gamma^2\left(\frac{1}{2}\right)}{\Gamma(1)} = 2\pi. \tag{122}$$

This trivial by itself result (obviously $I_{0,0} = \int_0^{2\pi}\mathrm{d}x = 2\pi$) confirms the correctness of Eq. (121).

Furthermore, $\Gamma\left(1+\frac{1}{2}\right) = \frac{1}{2}\Gamma\left(\frac{1}{2}\right) = \frac{1}{2}\sqrt{\pi}$, $\Gamma(2) = 1$, $\Gamma(3) = 2\Gamma(2) = 2$, thus

$$I_{2,0} = \frac{2\Gamma\left(1+\frac{1}{2}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma(2)} = \Gamma^2\left(\frac{1}{2}\right) = \pi \tag{123}$$

and

$$I_{2,2} = \frac{2\Gamma^2\left(1+\frac{1}{2}\right)}{\Gamma(3)} = \frac{\pi}{4}. \tag{124}$$

Finally, $\Gamma\left(2+\frac{1}{2}\right) = \frac{3}{2}\Gamma\left(\frac{3}{2}\right) = \frac{3}{4}\sqrt{\pi}$, and

$$I_{4,0} = \frac{2\Gamma\left(2+\frac{1}{2}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma(3)} = \frac{3\pi}{4}. \tag{125}$$

References

[1] Steven H. Strogatz. *Nonlinear Dynamics and Chaos: with Applications to Physics, Biology, Chemistry, and Engineering*. Westview Press, 2001.

[2] Carl M. Bender and Steven A. Orszag. *Advanced Mathematical Methods for Scientists and Engineers*. Springer Verlag, 1999.

[3] N.N. Bogoliubov and Y.A. Mitropolsky. *Asymptotic Methods in the Theory of Non-linear Oscillations*. Gordon and Breach, 1961.