THE METHOD OF AVERAGING

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https://www.phys.uconn.edu/~rozman/Courses/P2400_25S/

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Let's consider the following second order non-linear differential equation

$$\frac{\mathrm{d}^2 x}{\mathrm{d}t^2} + \varepsilon \left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^3 + x = 0, \quad \varepsilon > 0 \tag{1}$$

with the initial conditions

$$x(0) = x_0\,, \qquad \dot{x}(0) = v_0\,. \tag{2}$$

The equation describes a non-liner oscillator with the "friction" force that is proportional to the third power of the velocity. The parameter ε is a positive parameter that describes the rate of the energy loss, $\mathrm{d}E/\mathrm{d}t$, in the system. Equation (1) has no exact analytic solutions.

To obtain an approximate analytic solution of Eq. (1), we use a powerful method called the *method of averaging*. It is applicable to equations of the following general form:

$$\frac{\mathrm{d}^2 x}{\mathrm{d}t^2} + \varepsilon F\left(x, \frac{\mathrm{d}x}{\mathrm{d}t}\right) + x = 0, \tag{3}$$

where in our case

$$F\left(x, \frac{\mathrm{d}x}{\mathrm{d}t}\right) = \left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^3. \tag{4}$$

We seek a solution to Eq. (3) in the form:

$$x = a(t)\cos\left(t + \psi(t)\right), \tag{5}$$

$$\frac{\mathrm{d}x}{\mathrm{d}t} = -a(t)\sin\left(t + \psi(t)\right). \tag{6}$$

The motivation for this ansatz is that when ε is zero, Eq. (3) has its solution of the form Eq. (5) with a and ψ constants. For small values of ε we expect the same form of the solution to be approximately valid, but now a and ψ are expected to be slowly varying functions of t.

Differentiating Eq. (5) and requiring Eq. (6) to hold, we obtain the following relation:

$$\frac{\mathrm{d}a}{\mathrm{d}t}\cos\left(t+\psi(t)\right) - a\frac{\mathrm{d}\psi}{\mathrm{d}t}\sin\left(t+\psi(t)\right) = 0. \tag{7}$$

Differentiation of Eq. (6) and substitution the result into Eq. (3) gives

$$-\frac{\mathrm{d}a}{\mathrm{d}t}\sin\left(t+\psi\right) - a\frac{\mathrm{d}\psi}{\mathrm{d}t}\cos\left(t+\psi\right) = -\varepsilon F\Big(a\cos\left(t+\psi\right), -a\sin\left(t+\psi\right)\Big), \tag{8}$$

where in our case

$$F\Big(a\cos\left(t+\psi\right), -a\sin\left(t+\psi\right)\Big) = -a^3\sin^3\left(t+\psi\right). \tag{9}$$

Solving Eqs. (7) and (8) for $\frac{\mathrm{d}a}{\mathrm{d}t}$ and $\frac{\mathrm{d}\psi}{\mathrm{d}t}$, we obtain the following system of two differential equations:

$$\begin{aligned} \frac{\mathrm{d}a}{\mathrm{d}t} &= \varepsilon F\Big(a\cos\left(t+\psi\right), -a\sin\left(t+\psi\right)\Big)\sin(t+\psi), & (10)\\ \frac{\mathrm{d}\psi}{\mathrm{d}t} &= \frac{\varepsilon}{a} F\Big(a\cos\left(t+\psi\right), -a\sin\left(t+\psi\right)\Big)\cos(t+\psi), & (11) \end{aligned}$$

or, specifically to our case,

$$\begin{aligned} \frac{\mathrm{d}a}{\mathrm{d}t} &= -\varepsilon a^3\sin^4\left(t+\psi\right) & (12)\\ \frac{\mathrm{d}\psi}{\mathrm{d}t} &= -\varepsilon a^2\sin^3\left(t+\psi\right)\cos(t+\psi). & (13) \end{aligned}$$

So far our treatment has been exact.

Now we introduce the following approximation: since ε is small, $\frac{\mathrm{d}a}{\mathrm{d}t}$ and $\frac{\mathrm{d}\psi}{\mathrm{d}t}$ are also small. Hence $a(t)$ and $\psi(t)$ are slowly varying functions of t. Thus over one cycle of oscillations the quantities $a(t)$ and $\psi(t)$ on the right hand sides of Eqs. (12) and (13) can be treated as nearly constant, and thus these right hand sides may be replaced by their averages:

$$\frac{1}{2\pi}\int_0^{2\pi}\mathrm{d}\phi\ldots, \tag{14}$$

where $\phi = t + \psi$.

Eqs. (12) and (13) become

$$\begin{aligned} \frac{\mathrm{d}a}{\mathrm{d}t} &= -\varepsilon\, a^3 \frac{1}{2\pi}\int_0^{2\pi} \mathrm{d}\phi\, \sin^4(\phi), && (15) \\ \frac{\mathrm{d}\psi}{\mathrm{d}t} &= -\varepsilon\, a^2 \frac{1}{2\pi}\int_0^{2\pi} \mathrm{d}\phi\, \sin^3(\phi)\cos(\phi). && (16) \end{aligned}$$

The right hand side of Eq. (16) is zero:

$$\int_0^{2\pi} \mathrm{d}\phi\, \sin^3(\phi)\cos(\phi) = \int_0^{2\pi} \sin^3(\phi)\, \mathrm{d}(\sin(\phi)) = \frac{1}{4}\sin^4(\phi)\Big|_0^{2\pi} = 0. \qquad (17)$$

The averaging in Eq. (15) can be done using the following trigonometric identity:

$$\sin^2(\phi) = \frac{1}{2}(1-\cos(2\phi)).$$

$$\frac{1}{2\pi}\int_0^{2\pi} \mathrm{d}\phi \cos^2(n\phi) = \frac{1}{2\pi}\int_0^{2\pi} \mathrm{d}\phi \sin^2(n\phi) = \frac{1}{2}, \quad n = 1, 2, \ldots$$

$$\frac{1}{2\pi}\int_0^{2\pi} \mathrm{d}\phi \cos(n\phi) = \frac{1}{2\pi}\int_0^{2\pi} \mathrm{d}\phi \sin(n\phi) = 0, \quad n = 1, 2, \ldots$$

$$\begin{aligned} \frac{1}{2\pi}\int_0^{2\pi} \mathrm{d}\phi \sin^4(\phi) &= \frac{1}{2\pi}\int_0^{2\pi} \mathrm{d}\phi \left(\frac{1}{2}(1-\cos(2\phi))\right)^2 = \\ &= \frac{1}{4}\frac{1}{2\pi}\int_0^{2\pi} \mathrm{d}\phi \left(1 - 2\cos(2\phi) + \cos^2(2\phi)\right) = \\ &= \frac{1}{4}\left(1 + \frac{1}{2}\right) = \frac{3}{8} && (18) \end{aligned}$$

The averaged equations are as following:

$$\frac{\mathrm{d}a}{\mathrm{d}t} = -\varepsilon\frac{3}{8}a^3 \tag{19}$$

$$\frac{\mathrm{d}\psi}{\mathrm{d}t} = 0 \tag{20}$$

The solution of Eq. (20) is

$$\psi = \psi_0 = \mathrm{const}, \tag{21}$$

where ψ_0 is determined by the initial conditions.

Eq. (19) can be solved by separating the variables:

$$\frac{\mathrm{d}a}{a^3} = -\frac{3}{8}\varepsilon \mathrm{d}t \quad \longrightarrow \quad \frac{1}{a^2(t)} = \frac{3}{4}\varepsilon t + \frac{1}{a_0^2} \quad \longrightarrow \quad a(t) = \frac{1}{\sqrt{\frac{3}{4}\varepsilon t + \frac{1}{a_0^2}}}, \tag{22}$$

where $a_0 = a(0)$ is the amplitude of oscillations at $t = 0$. Finally,

$$x(t) = \frac{\cos(t+\psi_0)}{\sqrt{\frac{3}{4}\varepsilon t + \frac{1}{a_0^2}}}, \qquad \dot{x}(t) = -\frac{\sin(t+\psi_0)}{\sqrt{\frac{3}{4}\varepsilon t + \frac{1}{a_0^2}}}. \tag{23}$$

The integration constants a_0 and ψ_0 are determined from the initial conditions,

$$x(0) = a_0\cos\psi_0, \qquad \dot{x}(0) = -a_0\sin\psi_0, \tag{24}$$

$$a_0 = \sqrt{x^2(0) + \dot{x}^2(0)}, \qquad \psi_0 = -\frac{\dot{x}(0)}{x(0)}. \tag{25}$$