REFLECTION FORMULA FOR GAMMA FUNCTION

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https://www.phys.uconn.edu/~rozman/Courses/P2400_24S/

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Gamma function satisfies the following identity for all complex z:

$$\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin(\pi z)}, \tag{1}$$

referred to as the Euler's *reflection formula*.

We start from the relation between gamma and beta functions:

$$B(x,y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}. \tag{2}$$

Choosing $x = z$, $y = 1-z$, i.e. $x+y=1$ and knowing that $\Gamma(1) = 1$, we obtain:

$$\Gamma(z)\Gamma(1-z) = B(z,1-z)\Gamma(1) = \int\limits_0^1 t^{z-1}(1-t)^{-z}\mathrm{d}t = \int\limits_0^1 \left(\frac{t}{1-t}\right)^z \frac{\mathrm{d}t}{t}, \tag{3}$$

where the first of the integrals is just the definition of beta function.

Perform the substitution

$$u^2 = \frac{t}{1-t}, \tag{4}$$

so that $0 \le u < \infty$ and

$$t = \frac{u^2}{1+u^2}, \quad \mathrm{d}t = \frac{2u\,\mathrm{d}u}{(1+u^2)^2}, \quad \frac{\mathrm{d}t}{t} = \frac{2\,\mathrm{d}u}{u\,(1+u^2)}. \tag{5}$$

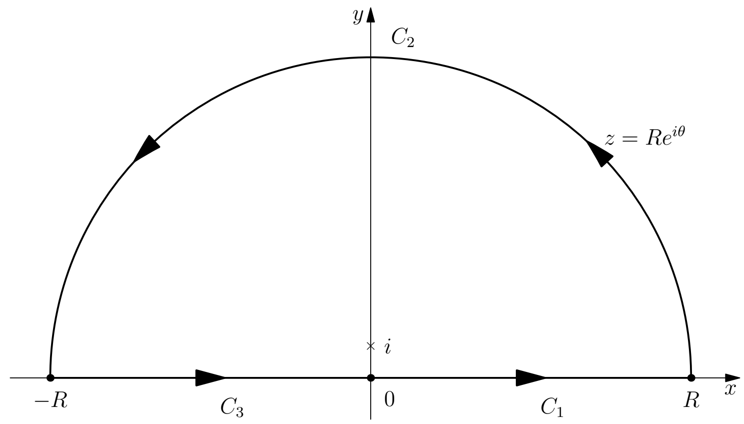

Figure 1: Integration contour in Eq. (8)

This transforms Eq. (3) into

$$\Gamma(z)\Gamma(1-z) = 2\int_0^\infty \frac{u^{2z-1}}{1+u^2}\,\mathrm{d}u. \tag{6}$$

To evaluate the integral in Eq. (6),

$$I = \int_0^\infty \frac{u^{2z-1}}{1+u^2}\,\mathrm{d}u, \tag{7}$$

consider the contour in the complex plane sketched in Fig. 1; we assume that $R \to \infty$.

$$J = \oint\limits_{C_1+C_2+C_3} \frac{w^{2z-1}}{1+w^2}\,\mathrm{d}w. \tag{8}$$

On the one hand, the integrand in Eq. (8) has a single simple pole inside the contour, at $w = i = e^{i\frac{\pi}{2}}$. Thus,

$$J = 2\pi i\,\mathrm{Res}\left(\frac{w^{2z-1}}{1+w^2}, w = e^{i\frac{\pi}{2}}\right) = 2\pi i\ \left.\frac{w^{2z-1}}{2w}\right|_{w=e^{i\frac{\pi}{2}}} = \pi\, e^{i\pi(z-\frac{1}{2})} = -i\pi\, e^{i\pi z}. \tag{9}$$

On the other hand,

$$J = \int\limits_{C_1} \frac{w^{2z-1}}{1+w^2}\,\mathrm{d}w + \int\limits_{C_2} \frac{w^{2z-1}}{1+w^2}\,\mathrm{d}w + \int\limits_{C_3} \frac{w^{2z-1}}{1+w^2}\,\mathrm{d}w = J_1 + J_2 + J_3. \tag{10}$$

Furthermore,

$$J_1 = \int\limits_{C_1} \frac{w^{2z-1}}{1+w^2}\,\mathrm{d}w = \int\limits_0^\infty \frac{u^{2z-1}}{1+u^2}\,\mathrm{d}u = I. \tag{11}$$

$$J_2 = 0. \tag{12}$$

On the contour C_3, $w = re^{i\pi}$, where r is a positive real variable changing from ∞ to 0.

$$J_3 = \int\limits_\infty^0 \frac{e^{i\pi(2z-1)}\,r^{2z-1}}{1+r^2}\,e^{i\pi}\,\mathrm{d}r = -e^{2i\pi z}\int\limits_0^\infty \frac{r^{2z-1}}{1+r^2}\,\mathrm{d}r = -e^{2i\pi z}I. \tag{13}$$

From Eqs. (9)–(13),

$$I\left(1 - e^{2i\pi z}\right) = -i\pi\,e^{i\pi z}, \tag{14}$$

$$I = \frac{i\pi}{e^{i\pi z} - e^{-i\pi z}} = \frac{\pi}{2\sin(\pi z)}. \tag{15}$$

Finally,

$$\Gamma(z)\Gamma(1-z) = 2I = \frac{\pi}{\sin(\pi z)}. \tag{16}$$

Raabe's formula and Euler's log-sine integral

We now use the reflection formula to evaluate the integral

$$I_R = \int\limits_0^1 \ln(\Gamma(x))\,\mathrm{d}x, \tag{17}$$

which is called the *Raabe's integral*.

We take the logarithm from both sides of Eq. (16) and integrate the result with respect to z from 0 to 1.

$$\int\limits_0^1 \ln(\Gamma(x))\,\mathrm{d}x + \int\limits_0^1 \ln(\Gamma(1-x))\,\mathrm{d}x = \ln\pi - \int\limits_0^1 \ln(\sin(\pi x))\,\mathrm{d}x. \tag{18}$$

The second integral in the left hand side of Eq. (18) is equal to the first one:

$$\int\limits_0^1 \ln(\Gamma(1-x))\,\mathrm{d}x = -\int\limits_1^0 \ln(\Gamma(t))\,\mathrm{d}t = \int\limits_0^1 \ln(\Gamma(t))\,\mathrm{d}t = I_R \tag{19}$$

where we introduced in Eq. (19) a new integration variable $t = 1 - x$, $1 \geq t \geq 0$, $\mathrm{d}x = -\mathrm{d}t$, and next swapped the integration limits.

We change the integration variable in the integral on the right of Eq. (18):

$$s = \pi x, \quad \mathrm{d}s = \pi \mathrm{d}x, \quad 0 \leq s \leq \pi,$$

$$\int_0^1 \ln(\sin(\pi x))\,\mathrm{d}x = \frac{1}{\pi}\int_0^\pi \ln(\sin(s))\,\mathrm{d}s. \tag{20}$$

Combining Eqs. (18)–(20), we obtain:

$$I_R = \frac{1}{2}\ln(\pi) - \frac{1}{2\pi}\int_0^\pi \ln(\sin(x))\,\mathrm{d}x \tag{21}$$

The integral in the right hand side of Eq. (21),

$$I_{\mathrm{ls}} = \int_0^\pi \ln(\sin(s))\,\mathrm{d}s, \tag{22}$$

is known as *Euler's log-sine integral*. It was first evaluated by Euler (published in 1769). Later in the course we evaluate the integral Eq. (22) using the contour integration. For now we resort to the following trick.

First, we change the integration range as following:

$$I_{\mathrm{ls}} = 2\int_0^{\pi/2} \ln(\sin(s))\,\mathrm{d}s. \tag{23}$$

Next, notice that

$$\int_0^{\pi/2} \ln(\sin(s))\,\mathrm{d}s = \int_0^{\pi/2} \ln(\cos(s))\,\mathrm{d}s. \tag{24}$$

Thus,

$$I_{\mathrm{ls}} = \int_0^{\pi/2} \ln(\sin(s))\,\mathrm{d}s + \int_0^{\pi/2} \ln(\cos(s))\,\mathrm{d}s = \int_0^{\pi/2} \ln(\sin(s)\cos(s))\,\mathrm{d}s. \tag{25}$$

Using the trigonometric identity

$$\sin(s)\cos(s) = \frac{1}{2}\sin(2s),$$

that is

$$\ln\left(\sin(s)\cos(s)\right) = -\ln 2 + \ln\left(\sin(2s)\right), \tag{26}$$

we get from Eq. (25):

$$I_{\mathrm{ls}} = -\frac{\pi}{2}\ln 2 + \int\limits_0^{\pi/2} \ln\left(\sin(2s)\right)\,\mathrm{d}s = -\frac{\pi}{2}\ln 2 + \frac{1}{2}\int\limits_0^{\pi} \ln\left(\sin(t)\right)\,\mathrm{d}t = -\frac{\pi}{2}\ln 2 + \frac{1}{2}I_{\mathrm{ls}}. \tag{27}$$

Finally,

$$I_{\mathrm{ls}} = \int\limits_0^{\pi} \ln\left(\sin(t)\right)\,\mathrm{d}t = -\pi\ln 2. \tag{28}$$

Returning to the Raabe's integral, Eq. (21), we have

$$\int\limits_0^1 \ln\left(\Gamma(x)\right)\,\mathrm{d}x = \frac{1}{2}\ln(\pi) - \frac{1}{2\pi}I_{\mathrm{ls}} = \frac{1}{2}\ln(\pi) + \frac{1}{2}\ln 2 = \frac{1}{2}\ln(2\pi). \tag{29}$$