LAPLACE'S METHOD FOR ORDINARY DIFFERENTIAL EQUATIONS

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https://www.phys.uconn.edu/~rozman/Courses/P2400_24S/

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It is possible to represent solutions of differential equations by definite integrals in which the independent variable appears as a parameter under the integral sign (see e.g. [1, pp. 124-8], [2, Ch. VIII], [3, Apps. a-b], [4, Ch. 18], [5, Ch. 8.A], [6, Ch. 5.3], [7, Ch. VII.46]). In this compact form properties of solutions to an equation become quite clear, asymptotic expansions can be obtained directly, and numerical computations may be facilitated.

One of the most important applications of this method is due to Laplace and affects the equation

$$(a_n + b_n x)\frac{\mathrm{d}^n y}{\mathrm{d}x^n} + (a_{n-1} + b_{n-1} x)\frac{\mathrm{d}^{n-1} y}{\mathrm{d}x^{n-1}} + \ldots + (a_0 + b_0 x)\, y = 0, \tag{1}$$

whose coefficients are at most of the first degree in x. Let us try to find a solution of this equation by taking for y an expression of the form

$$y(x) = \int_C Z(t)\, e^{xt}\, \mathrm{d}t, \tag{2}$$

where $Z(t)$ is a function of the variable t and where C is an unspecified yet integration contour independent of x. We have,

$$\frac{\mathrm{d}^p y}{\mathrm{d}x^p} = \int_C Z(t)\, t^p\, e^{xt}\, \mathrm{d}t, \tag{3}$$

and, replacing $y(x)$ and its derivatives in the left-hand side of Eq. (1) with Eq. (3), we find

$$\int_C Z(t)\, e^{xt}\, (P(t) + x\, Q(t))\, \mathrm{d}t = 0. \tag{4}$$

Here we introduced the notation

$$\begin{aligned} P(t) &= a_n t^n + a_{n-1} t^{n-1} + \ldots + a_0, & (5)\\ Q(t) &= b_n t^n + b_{n-1} t^{n-1} + \ldots + b_0. & (6) \end{aligned}$$

Integrating Eq. (4) by parts, we get

$$\begin{aligned} 0 &= \int_C Z(t)\,[P(t) + x\, Q(t)]\, e^{xt}\, \mathrm{d}t & (7)\\ &= \int_C Z(t)\, P(t)\, e^{xt}\, \mathrm{d}t + \int_C Z(t)\, Q(t)\, \mathrm{d}e^{xt} & (8)\\ &= \int_C \left(P(t)\, Z(t) - \frac{\mathrm{d}}{\mathrm{d}t}[Q(t)\, Z(t)]\right) e^{xt}\, \mathrm{d}t + \Big[Q(t)\, Z(t)\, e^{xt}\Big]_1^2 & (9) \end{aligned}$$

where the last term in Eq. (9) is evaluated at the end points of the contour C. If we choose the contour so as to make this contribution vanish,

$$\Big[Q(t)\, Z(t)\, e^{xt}\Big]_1^2 = 0, \tag{10}$$

then Eq. (2) will represent a solution to Eq. (1) if the function $Z(t)$ satisfies the differential equation

$$\frac{\mathrm{d}}{\mathrm{d}t}[Q(t)\, Z(t)] - P(t)\, Z(t) = 0. \tag{11}$$

Eq. (11) is the first order linear ordinary differential equation that can be solved separating variables:

$$\mathrm{d}\,[Q(t)\, Z(t)] = P(t)\, Z(t)\mathrm{d}t, \tag{12}$$

$$\frac{\mathrm{d}\,[Q(t)\, Z(t)]}{Q(t)\, Z(t)} = \frac{P(t)}{Q(t)}\, \mathrm{d}t, \tag{13}$$

$$\mathrm{d}\ln(Q(t)\, Z(t)) = \frac{P(t)}{Q(t)}\, \mathrm{d}t, \tag{14}$$

$$\ln(Q(t)\, Z(t)) = \int \frac{P(t)}{Q(t)}\, \mathrm{d}t + \alpha_1, \tag{15}$$

where α_1 is an integration constant. Exponentiating,

$$Q(t)\,Z(t) = \alpha \exp\left(\int \frac{P(t)}{Q(t)}\,\mathrm{d}t\right), \tag{16}$$

where $\alpha = \exp(\alpha_1)$ is another constant;

$$Z(t) = \frac{\alpha}{Q(t)} \exp\left(\int \frac{P(t)}{Q(t)}\,\mathrm{d}t\right). \tag{17}$$

Using Eq. (17) it is possible to determine suitable integration contour(s) to fulfill the requirement of Eq. (10).

Example 1. A boundary-value problem

Find the solution of the following boundary value problem:

$$x\frac{\mathrm{d}^3 y}{\mathrm{d}x^3} + 2y = 0, \tag{18}$$

$$y(0) = 1, \quad y(\infty) = 0. \tag{19}$$

Equation (18) is of Laplace's type. Following the general method, we identify the coefficients a_i and b_i, and form the functions $P(t)$, $Q(t)$, and $Z(t)$:

$$a_3 = 0, \quad b_3 = 1, \quad a_2 = 0, \quad b_2 = 0, \quad a_1 = 0, \quad b_1 = 0, \quad a_0 = 2, \quad b_0 = 0, \tag{20}$$

$$P(t) = a_3 t^3 + a_2 t^2 + a_1 t + a_0 = 2, \tag{21}$$

$$Q(t) = b_3 t^3 + b_2 t^2 + b_1 t + b_0 = t^3, \tag{22}$$

next

$$\int \frac{P(t)}{Q(t)}\,\mathrm{d}t = 2\int \frac{\mathrm{d}t}{t^3} = -\frac{1}{t^2}, \tag{23}$$

and finally

$$Z = \frac{\alpha}{t^3} \exp\left(-\frac{1}{t^2}\right). \tag{24}$$

The definite integral

$$y(x) = \int_C e^{xt}\,Z(t)\,\mathrm{d}t = \alpha \int_C \frac{e^{xt-\frac{1}{t^2}}}{t^3}\,\mathrm{d}t \tag{25}$$

is a particular solution of Eq. (18) if the function

$$Q(t)\,Z(t)\,e^{xt} = e^{xt-\frac{1}{t^2}} \tag{26}$$

takes on the same values at the ends of the path of integration.

Let's assume that $x > 0$ and choose the integration contour along the negative real axis, $-\infty < t \le 0$. Then,

$$Q(-\infty)\,Z(-\infty)\,\exp(-\infty) = Q(0)\,Z(0)\,\exp(0) = 0, \tag{27}$$

as required, and

$$y(x) = -\frac{\alpha}{2}\int_{-\infty}^{0} e^{xt-\frac{1}{t^2}}\,\mathrm{d}\left(\frac{1}{t^2}\right) = \int_{0}^{\infty} e^{-\frac{x}{\sqrt{u}}-u}\,\mathrm{d}u, \tag{28}$$

where we changed the integration variable to $u = \frac{1}{t^2}$, $0 \le u < \infty$, $t = -\frac{1}{\sqrt{u}}$, and chose the integration constant $\alpha = -2$ to satisfy the boundary condition $y(0) = 1$. Indeed, from Eq. (28),

$$y(0) = \int_{0}^{\infty} e^{-u}\,\mathrm{d}u = 1. \tag{29}$$

Let's verify that the solution Eq. (28) satisfies Eq. (18). Indeed,

$$\begin{aligned} x\frac{\mathrm{d}^3 y}{\mathrm{d}x^3} &= x\frac{\mathrm{d}^3}{\mathrm{d}x^3}\int_{0}^{\infty} e^{-\frac{x}{\sqrt{u}}-u}\,\mathrm{d}u = x\int_{0}^{\infty}\left(\frac{\mathrm{d}^3}{\mathrm{d}x^3}e^{-\frac{x}{\sqrt{u}}}\right)e^{-u}\,\mathrm{d}u = -x\int_{0}^{\infty} e^{-u}\,e^{-\frac{x}{\sqrt{u}}}\,\frac{\mathrm{d}u}{u^{\frac{3}{2}}} \\ &= 2x\int_{0}^{\infty} e^{-u}\,e^{-\frac{x}{\sqrt{u}}}\,\mathrm{d}\left(\frac{1}{\sqrt{u}}\right) = 2\int_{0}^{\infty} e^{-u}\,e^{-\frac{x}{\sqrt{u}}}\,\mathrm{d}\left(\frac{x}{\sqrt{u}}\right) = -2\int_{0}^{\infty} e^{-u}\,\mathrm{d}\left(e^{-\frac{x}{\sqrt{u}}}\right) \\ &= -2\left[e^{-u}e^{-\frac{x}{\sqrt{u}}}\right]_0^{\infty} - 2\int_{0}^{\infty} e^{-\frac{x}{\sqrt{u}}-u}\,\mathrm{d}u = -2\int_{0}^{\infty} e^{-\frac{x}{\sqrt{u}}-u}\,\mathrm{d}u = -2\,y(x). \end{aligned} \tag{30}$$

To find the behavior of $y(x)$ for large x we use the *Laplace's method* for integrals – a technique for obtaining the asymptotic behavior of integrals in which the large parameter appears in an exponential. The method relies on the observation that if the integrand has a maximum then for large x this maximum is very sharp. Then it is only the immediate neighborhood of the maximum that contributes to the asymptotic expansion of the integral for large x.

For the integrand in Eq. (28) the maximum occurs when

$$\frac{\mathrm{d}}{\mathrm{d}u}\left(-\frac{x}{\sqrt{u}}-u\right)=0 \quad\longrightarrow\quad \frac{x}{2\,u^{\frac{3}{2}}}-1=0 \quad\longrightarrow\quad u=\left(\frac{x}{2}\right)^{\frac{2}{3}}. \tag{31}$$

Such a maximum is called a *movable maximum* because its location depends on x.

For this kind of movable maximum problem, Laplace's method can be applied if we first transform the movable maximum to a *fixed maximum*. This is done by making the change of variables

$$u=x^{\frac{2}{3}}\,v, \quad \frac{x}{\sqrt{u}}=\frac{x^{\frac{2}{3}}}{\sqrt{v}}, \quad \mathrm{d}u=x^{\frac{2}{3}}\,\mathrm{d}v. \tag{32}$$

Then,

$$y(x)=x^{\frac{2}{3}}\int_0^\infty e^{-x^{\frac{2}{3}}\left(\frac{1}{\sqrt{v}}+v\right)}\,\mathrm{d}v=x^{\frac{2}{3}}\int_0^\infty e^{-x^{\frac{2}{3}}f(v)}\,\mathrm{d}v, \tag{33}$$

where in the exponent we introduce the notation

$$f(v)=\frac{1}{\sqrt{v}}+v. \tag{34}$$

The main contribution to the integral Eq. (33) comes from values of v in the small vicinity of the minimum of $f(v)$. The condition

$$\frac{\mathrm{d}f}{\mathrm{d}v}=-\frac{1}{2}v^{-\frac{3}{2}}+1=0 \tag{35}$$

gives the position of the minimum at $v_0=2^{-\frac{2}{3}}$. Expanding $f(v)$ into Taylor series near v_0 and keeping up to quadratic terms, we obtain:

$$f(v)\approx 3\left(2^{-\frac{2}{3}}+2^{-\frac{4}{3}}(v-v_0)^2\right). \tag{36}$$

Replacing in Eq. (33) $f(v)$ with its approximation Eq. (36),

$$y(x)\approx x^{\frac{2}{3}}e^{-3x^{\frac{2}{3}}2^{-\frac{2}{3}}}\int_0^\infty e^{-3x^{\frac{2}{3}}2^{-\frac{4}{3}}(v-v_0)^2}\,\mathrm{d}v\approx x^{\frac{2}{3}}e^{-3x^{\frac{2}{3}}2^{-\frac{2}{3}}}\int_{-\infty}^\infty e^{-3x^{\frac{2}{3}}2^{-\frac{4}{3}}w^2}\,\mathrm{d}w, \tag{37}$$

where we introduced a new integration variable $w = v - v_0$ and extended the lower integration limit to $-\infty$. The last integral is a gaussian one,

$$\int_{-\infty}^{\infty} e^{-3x^{\frac{2}{3}}2^{-\frac{4}{3}}w^2}\,\mathrm{d}w = \sqrt{\frac{\pi}{3x^{\frac{2}{3}}2^{-\frac{4}{3}}}} = x^{-\frac{1}{3}}2^{\frac{2}{3}}\sqrt{\frac{\pi}{3}}. \tag{38}$$

Therefore,

$$y(x) \approx 2\sqrt{\frac{\pi}{3}}\left(\frac{x}{2}\right)^{\frac{1}{3}} e^{-3\left(\frac{x}{2}\right)^{\frac{2}{3}}}. \tag{39}$$

The agreement between the approximation Eq. (39) numerical solution of differential equation (18) is illustrated in Fig. 1.

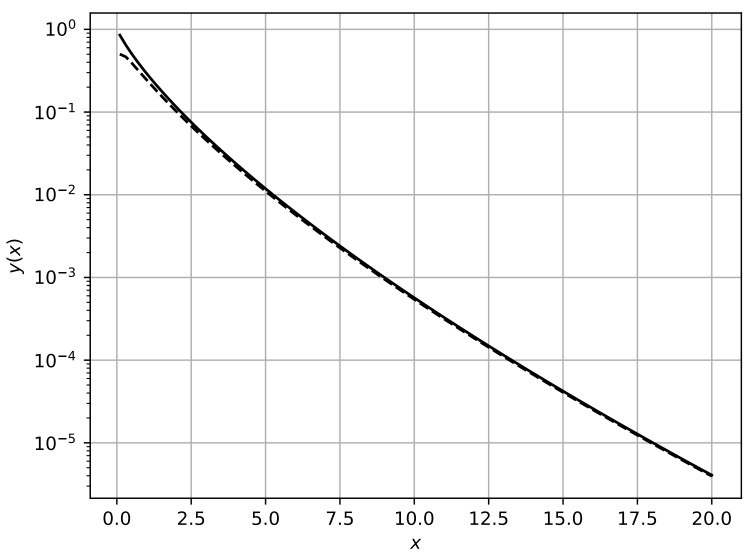

Figure 1: Asymptotics Eq. (39) (solid line) compared to the numerically evaluated integral (28) (dashed line) for $2 \le x \le 20$.

Example 2. Summation of a series

Find the behavior of the following sum for large positive values of the argument:

$$S(x) = \sum_{n=0}^{\infty} \frac{x^n}{(n!)^2} = 1 + x + \frac{x^2}{4} + \frac{x^3}{36} + \ldots. \tag{40}$$

To obtain the large x behavior of $S(x)$ we first construct a second-order differential equation satisfied by $S(x)$. Observe that

$$\frac{\mathrm{d}S}{\mathrm{d}x} = \sum_{n=1}^{\infty} \frac{x^{n-1}}{n!\,(n-1)!}, \tag{41}$$

$$x\frac{\mathrm{d}S}{\mathrm{d}x} = \sum_{n=1}^{\infty} \frac{x^{n}}{n!\,(n-1)!}, \tag{42}$$

and

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(x\frac{\mathrm{d}S}{\mathrm{d}x}\right) = \sum_{n=1}^{\infty} \frac{x^{n-1}}{((n-1)!)^2} = \sum_{m=0}^{\infty} \frac{x^{m}}{(m!)^2} = S(x). \tag{43}$$

Thus $S(x)$ is a solution to

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(x\frac{\mathrm{d}S}{\mathrm{d}x}\right) = S(x), \tag{44}$$

or

$$x\frac{\mathrm{d}^2S}{\mathrm{d}x^2} + \frac{\mathrm{d}S}{\mathrm{d}x} - S = 0. \tag{45}$$

We need to supplement Eq. (45) by two boundary conditions. For $x = 0$ the series Eq. (40) gives

$$S(0) = 1. \tag{46}$$

The coefficients in the series Eq. (40) are all positive, hence $S(x)$ is an increasing function of x. Thus

$$S(\infty) = \infty. \tag{47}$$

Equation (45) is of Laplace's type. Following the general method, we form the functions $P(t)$ and $Q(t)$:

$$a_2 = 0, \quad b_2 = 1, \quad a_1 = 1, \quad b_1 = 0, \quad a_0 = -1, \quad b_0 = 0, \tag{48}$$

$$P(t) = a_2t^2 + a_1t + a_0 = t - 1, \quad Q(t) = b_2t^2 + b_1t + b_0 = t^2, \tag{49}$$

$$\int \frac{P(t)}{Q(t)}\,\mathrm{d}t = \int \left(\frac{1}{t} - \frac{1}{t^2}\right)\mathrm{d}t = \log t + \frac{1}{t}, \tag{50}$$

and

$$Z(t) = \frac{\alpha}{Q(t)}\exp\left(\int \frac{P(t)}{Q(t)}\,\mathrm{d}t\right) = \frac{\alpha}{t}\exp\left(\frac{1}{t}\right), \tag{51}$$

where α is a yet unspecified integration constant.

The definite integral

$$S(x) = \int_C Z(t)\, e^{xt}\, \mathrm{d}t = \alpha \int_C \frac{e^{xt+\frac{1}{t}}}{t}\, \mathrm{d}t \tag{52}$$

is therefore a particular integral of Eq. (45) if we chose a closed contour C or if the function

$$Q(t)\, Z(t)\, e^{xt} = t\, e^{xt+\frac{1}{t}} \tag{53}$$

takes on the same values at the extremities of the integration contour C.

Let's write Eq. (52) in a more symmetric form by introducing a new integration variable, τ,

$$\tau = \sqrt{x}\, t, \quad t = \frac{\tau}{\sqrt{x}}, \quad \frac{1}{t} = \frac{\sqrt{x}}{\tau}, \quad t\, x = \tau \sqrt{x}, \quad \frac{\mathrm{d}t}{t} = \frac{\mathrm{d}\tau}{\tau}. \tag{54}$$

Then,

$$S(x) = \alpha \int_C e^{\sqrt{x}\left(\tau+\frac{1}{\tau}\right)}\, \frac{\mathrm{d}\tau}{\tau}. \tag{55}$$

To satisfy Eq. (46) and Eq. (47) we choose the integration contour to be a closed loop around the origin:

$$\tau = e^{i\theta}, \quad 0 \le \theta \le 2\pi, \quad \frac{\mathrm{d}\tau}{\tau} = i\, \mathrm{d}\theta, \quad \tau + \frac{1}{\tau} = 2\cos\theta \tag{56}$$

$$S(x) = i\, \alpha \int_0^{2\pi} e^{2\sqrt{x}\cos\theta}\, \mathrm{d}\theta. \tag{57}$$

To satisfy the boundary condition Eq. (46), we chose $\alpha = -\frac{i}{2\pi}$. Finally,

$$S(x) = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{2\sqrt{x}\cos\theta}\, \mathrm{d}\theta, \tag{58}$$

where we also shifted the integration limits for symmetry.

For small values of x,

$$\begin{aligned} S(x) &= \frac{1}{2\pi} \int_{-\pi}^{\pi} \left(1 + 2\sqrt{x}\cos\theta + 2\, x \cos^2\theta + \frac{4}{3} x^{\frac{3}{2}} \cos^3\theta + \frac{2}{3} x^2 \cos^4\theta + \ldots\right) \mathrm{d}\theta \quad (59) \\ &= \frac{1}{2\pi} \int_{-\pi}^{\pi} \mathrm{d}\theta + \frac{x}{\pi} \int_{-\pi}^{\pi} \cos^2\theta\, \mathrm{d}\theta + \frac{x^2}{3\pi} \int_{-\pi}^{\pi} \cos^4\theta\, \mathrm{d}\theta + \ldots \quad (60) \end{aligned}$$

where to get Eq. (59) we expanded $e^{2\sqrt{x}\cos\theta}$ into Taylor series. To get Eq. (60) we use the fact that the averaged values of odd powers of cosine over its period are all zero. Evaluating trigonometric integral in Eq. (60), we obtain

$$S(x) = 1 + x + \frac{x^2}{4} + \dots \tag{61}$$

which is indeed the expansion Eq. (40).

To find the behavior of $S(x)$ for large x we use the Laplace method for integrals. The main contribution to the integral Eq. (58) comes from small values of θ where $\cos\theta \approx 1 - \frac{\theta^2}{2} + \frac{\theta^4}{24}$. Therefore,

$$S(x) \approx \frac{e^{2\sqrt{x}}}{2\pi} \int_{-\infty}^{\infty} e^{-\sqrt{x}\theta^2} \left(1 + \frac{\sqrt{x}\theta^4}{12}\right) \mathrm{d}\theta = \frac{e^{2\sqrt{x}}}{2\sqrt{\pi\sqrt{x}}} \left(1 + \frac{1}{16\sqrt{x}}\right). \tag{62}$$

The agreement between the approximation Eq. (62) numerically calculated sum of Eq. (40) is illustrated in Fig. 2.

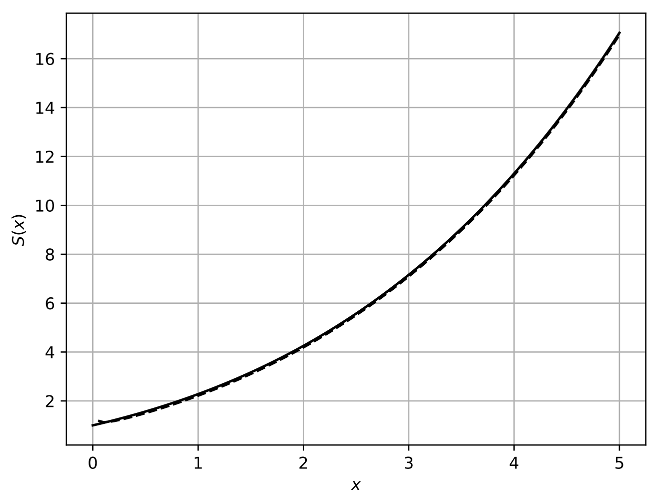

Figure 2: Asymptotics Eq. (62) (solid line) compared to the numerically evaluated sum (40) (dashed line) for $1 \le x \le 5$.

Example 3. Integral equation

Find the finite everywhere solution of the following integral equation:

$$y(x) = \int_{x}^{\infty} s\,(s - x)\,y(s)\,\mathrm{d}s. \tag{63}$$

Let's differentiate the integral equation two times to form a homogeneous second order linear differential equation. Using Leibniz formula,

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}}{\mathrm{d}x}\left\{\int_x^\infty s\,(s-x)\,y(s)\,\mathrm{d}s\right\} = -\,s\,y(s)\,(s-x)\Big|_{s=x} - \int_x^\infty s\,y(s)\,\mathrm{d}s = -\int_x^\infty s\,y(s)\,\mathrm{d}s. \tag{64}$$

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = \frac{\mathrm{d}}{\mathrm{d}x}\left\{\frac{\mathrm{d}y}{\mathrm{d}x}\right\} = -\frac{\mathrm{d}}{\mathrm{d}x}\left\{\int_x^\infty s\,y(s)\,\mathrm{d}s\right\} = s\,y(s)\Big|_{s=x} = x\,y(x). \tag{65}$$

Thus, we need to solve the following Laplace's type equation:

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} - xy = 0. \tag{66}$$

Equation (66) is called Airy equation. Its finite everywhere solution is called Airy function, $\mathrm{Ai}(x)$.

Following the general method of solving Laplace's type ODEs , we form the functions $P(t)$ and $Q(t)$:

$$a_0 = 1, \quad b_0 = 0, \quad a_1 = 0, \quad b_1 = 0, \quad a_2 = 0, \quad b_2 = -1, \tag{67}$$

$$P(t) = a_0t^2 + a_1t + a_2 = t^2, \qquad Q(t) = b_0t^2 + b_1t + b_2 = -1, \tag{68}$$

$$\int \frac{P(t)}{Q(t)}\,\mathrm{d}t = -\frac{t^3}{3}, \quad Z(t) = -e^{-\frac{t^3}{3}}, \tag{69}$$

so that the solution can be represented in the form

$$y(x) = \int_C e^{xt-\frac{t^3}{3}}\,\mathrm{d}t. \tag{70}$$

The path of integration C must be chosen so that the function

$$e^{xt-\frac{t^3}{3}} \tag{71}$$

vanishes at both ends of it. These ends must therefore go to infinity in the regions of the complex plane of t in which $\mathrm{Re}\,t^3 > 0$.

The solution finite for all x is obtained by taking the path as shown in Figure 3. It can be displaced in any manner provided that the ends of it go to infinity in the same two sectors.

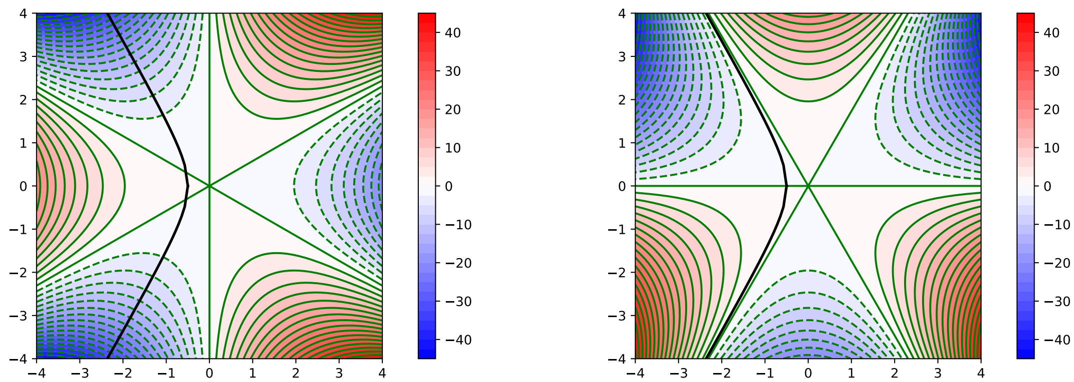

Figure 3: Contour lines of $\mathrm{Re}\left(-\frac{z^3}{3}\right)$ left, and $\mathrm{Im}\left(-\frac{z^3}{3}\right)$ right. The bold line is the same integration contour shown on both graphs.

Deforming the path so that it goes along the imaginary axis, we obtain the function in the form (substituting $t = iu$)

$$\mathrm{Ai}(x) = \frac{1}{\pi}\int_0^\infty \cos\left(ux + \frac{u^3}{3}\right)\mathrm{d}u. \tag{72}$$

The choice of the coefficient $\frac{1}{\pi}$ in Eq. (72) follows the conventional definition of Airy function.

For reference,

$$\mathrm{Ai}(0) = \frac{1}{\pi}\int_0^\infty \cos\left(\frac{u^3}{3}\right)\mathrm{d}u = \frac{\Gamma\left(\frac{1}{3}\right)}{3^{\frac{1}{6}}2\pi}. \tag{73}$$

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