LAPLACE'S METHOD FOR INTEGRALS

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https://www.phys.uconn.edu/~rozman/Courses/P2400_25S/

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Laplace's method is a general technique for obtaining the asymptotic behavior of integrals in which the large parameter λ, $\lambda \to \infty$, appears in the exponent:

$$I(\lambda) = \int_a^b f(t)\, e^{\lambda \phi(t)}\, \mathrm{d}t = \int_a^b f(t) \left(e^{\phi(t)}\right)^{\lambda} \mathrm{d}t. \tag{1}$$

Here $f(t)$ and $\phi(t)$ are real continuous functions, independent of λ. Integrals of this form are called *Laplace integrals*. Laplace's method relies on the following observation: if the real continuous function $\phi(t)$ has its maximum on the interval $a \le t \le b$ at $t = t_0$ and if $f(t_0) \neq 0$, then it is only the immediate neighborhood of $t = t_0$ that contributes to the asymptotic expansion of $I(\lambda)$ for large λ.

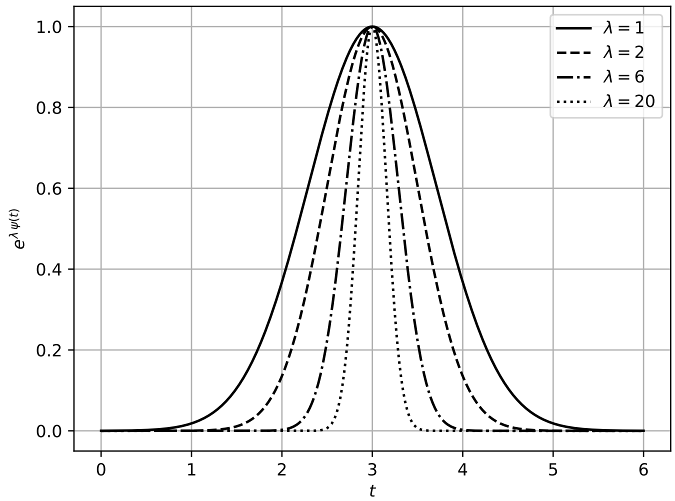
Indeed, we can always write $e^{\phi(t)}$ as $e^{\phi(t_0)}\, e^{\psi(t)}$, where $e^{\phi(t_0)}$ is just a constant multiplication factor that can be factored out of the integral. Here we defined $\psi(t) \equiv \phi(t) - \phi(t_0)$. The maximal value of $\psi(t)$ is zero, thus the maximal value of $e^{\psi(t)}$ is one. A typical behavior of $e^{\psi(t)}$ is sketched in Fig. 1 in solid line. As we rise $e^{\psi(t)}$ into power λ, its maximum stays "fixed" at $(x_0, 1)$ but its wings are "moving down" toward the x axis, thus making the graph narrower (see Fig. 1). Therefore we can replace $f(t)$ and $\phi(t)$ with their approximations that need to be good ones only in the vicinity of t_0.

The logic of the Laplace method works without changes for a more general form of the integrand:

$$\int_a^b f(t) \left(\kappa(t)\right)^{\lambda} \mathrm{d}t, \tag{2}$$

where λ is a large parameter as before, $\lambda \to \infty$, $\kappa(t)$ is real continuous function that is

Figure 1: Changes of the integrand in Laplace integral as the parameter λ is increasing.



independent of λ, doesn't change the sign on the interval of integration $a \le t \le b$ (say it is positive), and has a single maximum on that interval.

Let's consider simple examples of Laplace's method.

Example 1. Find the leading term of the asymptotics of the following integral for $\lambda \to \infty$:

$$I(\lambda) = \int_0^{\frac{\pi}{2}} \frac{e^{\lambda \cos(x)}}{x^2 + 4} \, dx. \tag{3}$$

Since only small $|x|$, such that $|x| \sim \frac{1}{\sqrt{\lambda}} \ll 1$, are important in the integral Eq. (11), we can approximate the integrand as following:

$$\cos(x) \approx 1 - \frac{x^2}{2}, \quad \longrightarrow \quad e^{\lambda \cos(x)} \approx e^{\lambda} e^{-\frac{\lambda}{2}x^2}, \tag{4}$$

$$\frac{1}{x^2 + 4} \approx \frac{1}{4}. \tag{5}$$

Thus,

$$I(\lambda) \sim \frac{1}{4} e^{\lambda} \int_0^{\infty} e^{-\frac{\lambda}{2}x^2} \, dx = \frac{1}{8}\sqrt{\frac{2\pi}{\lambda}} e^{\lambda} = \boxed{\sqrt{\frac{\pi}{32\,\lambda}} e^{\lambda}}. \tag{6}$$

The agreement between the approximation Eq. (6) and the numerically evaluated integral Eq. (3) is shown in Fig. 2.

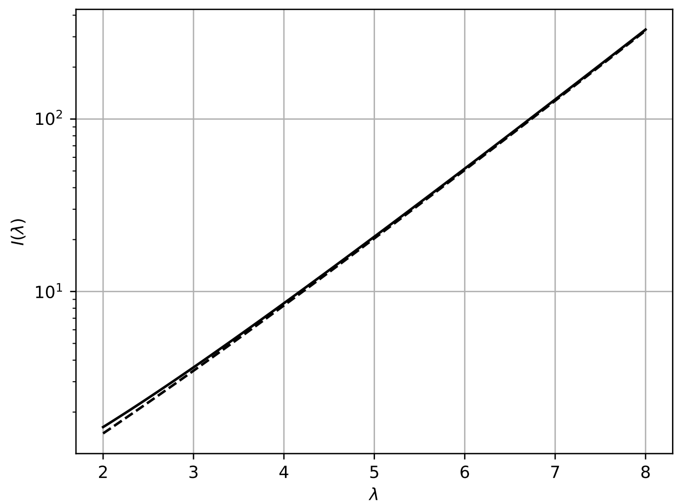


Figure 2: Asymptotics Eq. (6) (solid line) compared to the numerically evaluated integral (3) (dashed line) for $2 \leq \lambda \leq 8$. Notice the logarithmic scale on y axis.

Example 2. Find the leading term of the asymptotics of the following integral for $\lambda \to \infty$:

$$I(\lambda) = \int_0^1 e^{-\lambda \sin^3(x)} \, dx. \tag{7}$$

The maximum of the function in the exponent, $e^{-\sin^3 x}$ is at $x = 0$, so in this example the main contribution to the integral is coming from the vicinity of the left endpoint of the integration range, $x = 0$, where $\sin^3 x \sim x^3$.

$$I(\lambda) \sim \int_0^\infty e^{-\lambda x^3} dx. \tag{8}$$

To evaluate the last integral, let's introduce a new integration variable, $u = \lambda x^3$:

$$x^3 = \frac{u}{\lambda} \quad \longrightarrow \quad x = \frac{u^{\frac{1}{3}}}{\lambda^{\frac{1}{3}}} \quad \longrightarrow \quad dx = \frac{u^{\frac{1}{3}-1}}{3\lambda^{\frac{1}{3}}} du. \tag{9}$$

$$I(\lambda) \sim \frac{1}{3\lambda^{\frac{1}{3}}} \int_0^\infty e^{-u} u^{\frac{1}{3}-1} \, du = \frac{\Gamma(\frac{1}{3})}{3\lambda^{\frac{1}{3}}} = \boxed{\frac{\Gamma(\frac{4}{3})}{\lambda^{\frac{1}{3}}}}. \tag{10}$$

The agreement between the approximation Eq. (10) and the numerically evaluated integral Eq. (7) is shown in Fig. 3.

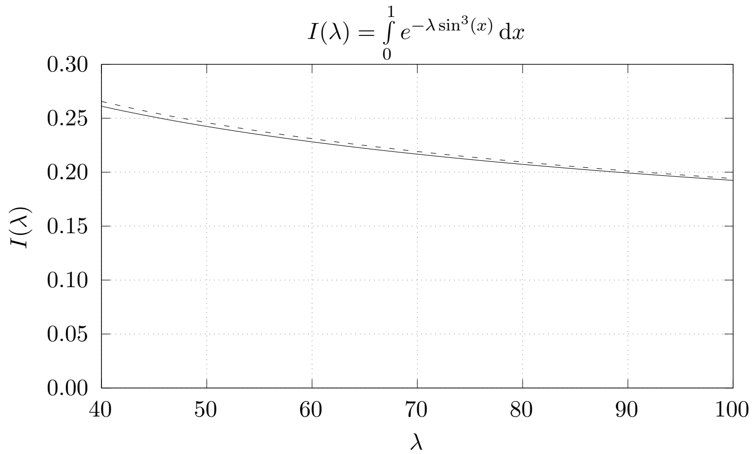


Figure 3: Asymptotics Eq. (10) (solid line) compared to the numerically evaluated integral (7) (dashed line) for $40 \le \lambda \le 100$.

Example 3. Find the leading term of the asymptotics of the following integral for $\lambda \to \infty$:

$$I(\lambda) = \int_{-3}^{4} e^{-\lambda x^2} \log\left(1 + x^2\right) \mathrm{d}x. \tag{11}$$

Since only small $|x|$, such that $|x| \sim \frac{1}{\sqrt{\lambda}} \ll 1$, are important in the integral Eq. (11) (for $|x| \ge \frac{1}{\sqrt{\lambda}}$ the integrand is negligibly small due to the exponent's factor), we can approximate the function in the integrand as following:

$$\log\left(1 + x^2\right) \sim x^2. \tag{12}$$

Thus,

$$I(\lambda) \sim \int_{-3}^{4} e^{-\lambda x^2} x^2 \,\mathrm{d}x \sim \int_{-\infty}^{\infty} e^{-\lambda x^2} x^2 \,\mathrm{d}x = 2\int_{0}^{\infty} e^{-\lambda x^2} x^2 \,\mathrm{d}x. \tag{13}$$

Introducing the new integration variable u,

$$u = \lambda x^2 \quad \longrightarrow \quad x^2 = \frac{u}{\lambda} \quad \longrightarrow \quad x = \frac{1}{\sqrt{\lambda}} u^{\frac{1}{2}} \quad \longrightarrow \quad \mathrm{d}x = \frac{1}{2\sqrt{\lambda}} u^{-\frac{1}{2}} \,\mathrm{d}u. \tag{14}$$

$$I(\lambda) \sim \lambda^{-\frac{3}{2}} \int_{0}^{\infty} e^{-u} u^{\frac{1}{2}} \,\mathrm{d}u = \lambda^{-\frac{3}{2}} \Gamma\left(\frac{3}{2}\right) = \lambda^{-\frac{3}{2}} \frac{1}{2} \Gamma\left(\frac{1}{2}\right) = \boxed{\frac{\sqrt{\pi}}{2} \lambda^{-\frac{3}{2}}} \tag{15}$$

The agreement between the approximation Eq. (15) and the numerically evaluated integral Eq. (11) is shown in Fig. 4.

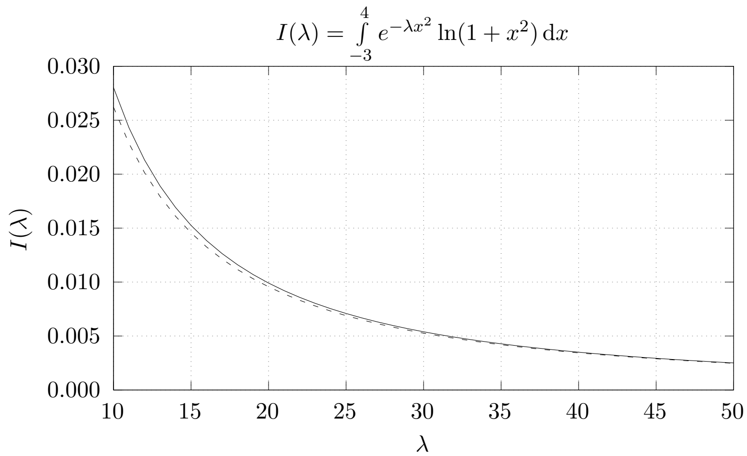


Figure 4: Asymptotics Eq. (15) (solid line) compared to the numerically evaluated integral Eq. (11) (dashed line) for $10 \le \lambda \le 50$.

Example 4. Find the leading term of the asymptotics of the following integral for $n \gg 1$:

$$I(n) = \int_{-1}^{1} (\cos x)^n \, dx, \tag{16}$$

Since only small $|x|$, such that $|x| \sim \frac{1}{\sqrt{\lambda}} \ll 1$, are important in the integral, we can approximate

$$\cos x \sim 1 - \frac{x^2}{2} \sim e^{-\frac{x^2}{2}}. \tag{17}$$

Thus,

$$I(n) = \int_{-1}^{1} \left(e^{-\frac{x^2}{2}}\right)^n dx \sim \int_{-\infty}^{\infty} e^{-\frac{nx^2}{2}} dx = \int_{-\infty}^{\infty} e^{-\left(\sqrt{\frac{n}{2}}x\right)^2} dx = \tag{18}$$

$$= \sqrt{\frac{2}{n}} \int_{-\infty}^{\infty} e^{-\left(\sqrt{\frac{n}{2}}x\right)^2} d\left(\sqrt{\frac{n}{2}}x\right) = \sqrt{\frac{2}{n}} \int_{-\infty}^{\infty} e^{-u^2} du = \tag{19}$$

$$= \boxed{\sqrt{\frac{2\pi}{n}}}. \tag{20}$$

The agreement between the approximation Eq. (20) and the numerically evaluated integral Eq. (16) is shown in Fig. 5.

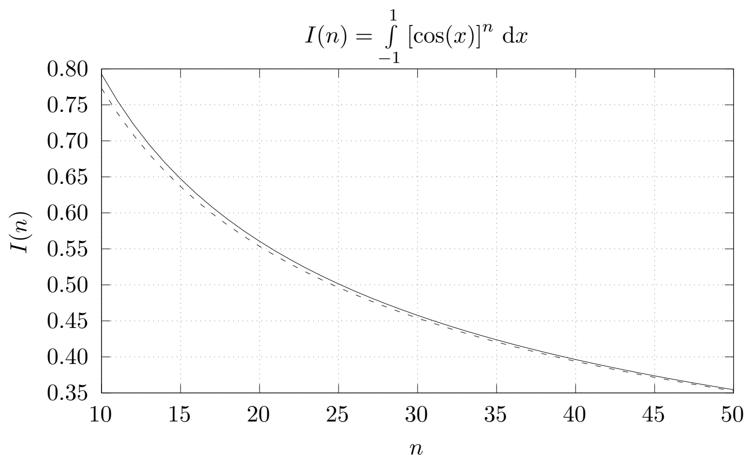


Figure 5: Asymptotics Eq. (20) (solid line) compared to the numerically evaluated integral (16) (dashed line) for $10 \le n \le 50$.

Example 5. Find the leading term of the asymptotics of gamma function, $\Gamma(x)$, for $x \to \infty$:

$$\Gamma(x) = \int_0^\infty e^{-t} t^{x-1}\, dt = \int_0^\infty e^{-t + x \log t} \frac{1}{t}\, dt \tag{21}$$

The function in the exponent in Eq. (21),

$$f(t) = -t + x \log t, \tag{22}$$

has its maximum at $t = t_0$ which depends upon x:

$$\frac{df}{dt} = 0, \quad \longrightarrow \quad -1 + \frac{x}{t} = 0 \quad \longrightarrow \quad t_0 = x. \tag{23}$$

To make the maximum independent of x, let's introduce a new integration variable, s,

$$s = \frac{t}{x}, \quad \longrightarrow \quad t = xs, \quad \longrightarrow \quad dt = x\, ds, \quad \frac{dt}{t} = \frac{ds}{s}, \tag{24}$$

$$f(t) = -t + x \log t = -xs + x \log s + x \log x. \tag{25}$$

$$\Gamma(x) = e^{x \log x} \int_0^\infty e^{-x(s - \log s)} \frac{1}{s}\, ds. \tag{26}$$

Let's apply the Laplace's method to the integral in Eq. (26):

$$f(s) = s - \log s, \quad \frac{df}{ds} = 1 - \frac{1}{s}. \tag{27}$$

$$\frac{\mathrm{d}f}{\mathrm{d}s} = 0, \quad \longrightarrow \quad s_0 = 1. \tag{28}$$

$$f(s_0) = 1, \quad \frac{\mathrm{d}^2 f}{\mathrm{d}s^2} = \frac{1}{s^2}, \quad \longrightarrow \quad \frac{\mathrm{d}^2 f}{\mathrm{d}s^2}(s_0) = 1. \tag{29}$$

$$f(s) \approx f(s_0) + \frac{1}{2}\frac{\mathrm{d}^2 f}{\mathrm{d}s^2}(s_0)(s-s_0)^2 = 1 + \frac{1}{2}(s-1)^2. \tag{30}$$

$$\int_0^\infty e^{-xf(s)}\frac{1}{s}\,\mathrm{d}s \sim \int_0^\infty e^{-x\left(1+\frac{1}{2}(s-1)^2\right)}\frac{1}{s_0}\,\mathrm{d}s \sim e^{-x}\int_{-\infty}^\infty e^{-\frac{1}{2}x(s-1)^2}\,\mathrm{d}s = e^{-x}\int_{-\infty}^\infty e^{-\frac{1}{2}xs^2}\,\mathrm{d}s. \tag{31}$$

The last integral is a Gaussian one:

$$\int_{-\infty}^\infty e^{-\frac{1}{2}xs^2}\,\mathrm{d}s = \sqrt{\frac{2\pi}{x}}, \tag{32}$$

therefore

$$\int_0^\infty e^{-x(s-\log s)}\frac{1}{s}\,\mathrm{d}s \sim e^{-x}\sqrt{\frac{2\pi}{x}}. \tag{33}$$

Finally, combining Eq. (26) and Eq. (33)

$$\Gamma(x) \sim e^{x\log x}e^{-x}\sqrt{\frac{2\pi}{x}} = \boxed{\sqrt{\frac{2\pi}{x}}\left(\frac{x}{e}\right)^x}. \tag{34}$$

The expression Eq. (34) is the leading term of the *Stirling approximation* for Gamma function.

The agreement between the approximation Eq. (34) and the numerically evaluated integral Eq. (21) is shown in Fig. 6.

References

[1] Carl M. Bender and Steven A. Orszag. *Advanced Mathematical Methods for Scientists and Engineers*. Springer Verlag, 1999.

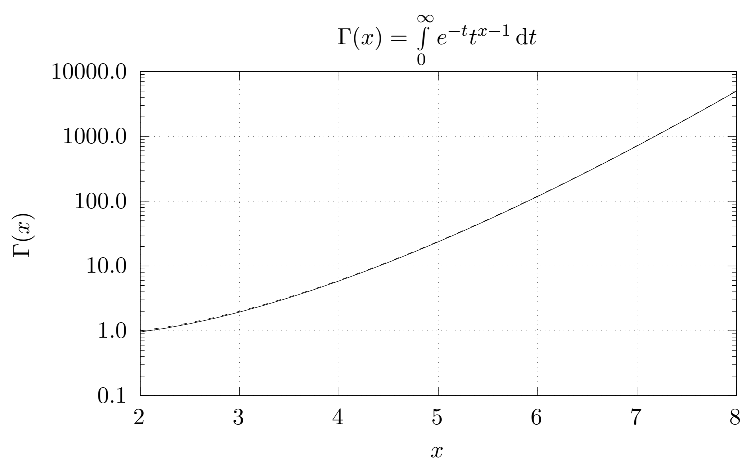


Figure 6: Asymptotics Eq. (34) (solid line) compared to the numerically evaluated integral (21) (dashed line) for $2 \le x \le 8$. Notice the logarithmic scale on y axis.