

JORDAN'S LEMMA

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https://www.phys.uconn.edu/~rozman/Courses/P2400_24S/

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When evaluating integrals using complex variables, we often need to show that

$$\lim_{R \rightarrow \infty} \int_{C_R} g(z) dz = 0, \quad (1)$$

where the integration contour C_R is a semicircular arc of radius R in the upper half-plane, $z = Re^{i\theta}$, $0 \leq \theta \leq \pi$, and $g(z)$ is an analytic function (except possibly for a finite number of poles).

Eq. (1) is true if $g(z)$ decreases $\sim \frac{1}{|z|^2}$ (or faster) as $z \rightarrow \infty$. Indeed, on a circular arc

$$dz = i R e^{i\theta} d\theta, \quad 0 \leq \theta \leq \pi. \quad (2)$$

The absolute value of integral Eq. (1),

$$\left| \int_{C_R} g(z) dz \right| = \left| i R \int_0^\pi g(Re^{i\theta}) d\theta \right| \leq R \int_0^\pi |g(Re^{i\theta})| d\theta \quad (3)$$

$$\leq R \frac{m}{R^2} \int_0^\pi d\theta \sim \frac{1}{R}. \quad (4)$$

Therefore, the absolute value of the integral and hence the integral itself are 0 as $R \rightarrow \infty$.

Jordan's Lemma extends this result for a special form of $g(z)$,

$$g(z) = f(z) e^{i\lambda z}, \quad \lambda > 0, \quad (5)$$

where λ is a real parameter, $\lambda > 0$, from functions $F(z)$ satisfying $f(z) \sim \frac{1}{|z|^2}$ as $|z| \rightarrow \infty$ to functions $f(z)$ satisfying $f(z) \rightarrow 0$ as $|z| \rightarrow \infty$. For $\lambda < 0$, the same conclusion holds for the semicircular contour C_R in the lower half-plane.

Indeed,

$$\int_{C_R} f(z) e^{i\lambda z} dz = iR \int_0^\pi f(Re^{i\theta}) e^{i(\lambda R \cos(\theta) + \theta)} e^{-\lambda R \sin(\theta)} d\theta. \quad (6)$$

Therefore,

$$\left| \int_{C_R} f(z) e^{i\lambda z} dz \right| = \left| iR \int_0^\pi f(Re^{i\theta}) e^{i(\lambda R \cos(\theta) + \theta)} e^{-\lambda R \sin(\theta)} d\theta \right| \quad (7)$$

$$\leq R \int_0^\pi \left| f(Re^{i\theta}) e^{i(\lambda R \cos(\theta) + \theta)} e^{-\lambda R \sin(\theta)} \right| d\theta \quad (8)$$

$$= R \int_0^\pi \left| f(Re^{i\theta}) \right| e^{-\lambda R \sin(\theta)} d\theta \quad (9)$$

$$\leq RM(R) \int_0^\pi e^{-\lambda R \sin(\theta)} d\theta \leq RM(R) \frac{\pi}{\lambda R} = \frac{\pi M(R)}{\lambda}. \quad (10)$$

Here

$$M(R) = \max_{0 \leq \theta \leq \pi} \left| f(Re^{i\theta}) \right|, \quad (11)$$

and we used that (see Fig. 1)

$$\int_0^\pi e^{-\lambda R \sin(\theta)} d\theta = 2 \int_0^{\frac{\pi}{2}} e^{-\lambda R \sin(\theta)} d\theta \leq 2 \int_0^{\frac{\pi}{2}} e^{-\lambda R \frac{2\theta}{\pi}} d\theta = \frac{\pi}{\lambda R} \int_0^{\frac{\pi}{2}} e^{-\frac{2\lambda R \theta}{\pi}} d\left(\frac{2\lambda R \theta}{\pi}\right) \quad (12)$$

$$= \frac{\pi}{\lambda R} \int_0^{\lambda R} e^{-u} du = \frac{\pi}{\lambda R} (1 - e^{-\lambda R}) \leq \frac{\pi}{\lambda R}. \quad (13)$$

Since $M(R) \rightarrow 0$ as $R \rightarrow \infty$, the absolute value of integral Eq. (6) and hence the integral itself are 0 as $R \rightarrow \infty$.

Figure 1: Illustration of the inequality $\sin(\theta) \geq \frac{2\theta}{\pi}$ for $0 \leq \theta \leq \frac{\pi}{2}$.

