$$\int_{-\infty}^{\infty} e^{-\frac{x^2}{2}} \cos(s+tx)\, dx = \int_{-\infty}^{\infty} e^{-\frac{x^2}{2}} \cos(s)\cos(tx)\, dx$$
$$- \int_{-\infty}^{\infty} e^{-\frac{x^2}{2}} \sin(s)\sin(tx)\, dx$$

where I've used the trig identity for the cosine of a sum, we have

$$\int_{-\infty}^{\infty} e^{-\frac{x^2}{2}} \cos(s+tx)\, dx = \cos(s)\int_{-\infty}^{\infty} e^{-\frac{x^2}{2}} \cos(tx)dx - \sin(s)\int_{-\infty}^{\infty} e^{-\frac{x^2}{2}} \sin(tx)dx.$$

But since the last integral on the right is zero because its integrand is odd, we have

$$\int_{-\infty}^{\infty} e^{-\frac{x^2}{2}} \cos(s+tx)\, dx = \cos(s)\int_{-\infty}^{\infty} e^{-\frac{x^2}{2}} \cos(tx)dx$$

and so, finally (using our result (3.1.5))

$$\int_{-\infty}^{\infty} e^{-\frac{x^2}{2}} \cos(s+tx)\, dx = \sqrt{2\pi}\, e^{-\frac{t^2}{2}} \cos(s)\,. \tag{3.1.6}$$

For $t=s=1$ this is 0.82144... and *quad*(@(*x*)*exp*(-(*x*.^2)/2).**cos*(*1* + *x*),-*10*,*10*)=0.82144...

The trick of evaluating an integral by finding a differential equation for which the integral is the solution can be used to determine the value of

$$I(a) = \int_0^{\infty} \frac{\cos(ax)}{x^2+b^2}\, dx$$

where a and b are each positive (a is the parameter and b is a constant). If we integrate by parts, writing

$$u = \frac{1}{x^2+b^2},\, dv = \cos(ax)dx$$

then

$$du = -\frac{2x}{\left(x^2+b^2\right)^2}dx,\, v = \frac{\sin(ax)}{a}$$

and so

$$I(a) = \left\{ \frac{\sin(ax)}{a\left(x^2+b^2\right)^2} \right\}\Big|_0^\infty + \frac{2}{a}\int_0^\infty \frac{x\sin(ax)}{\left(x^2+b^2\right)^2}\,dx$$

or, as the first term on the right vanishes at both the upper and the lower limit, we have

$$I(a) = \frac{2}{a}\int_0^\infty \frac{x\sin(ax)}{\left(x^2+b^2\right)^2}\,dx$$

and so it perhaps looks as though we are making things worse! As you'll soon see, we are not.

From our last result we multiply through by a and arrive at

$$aI(a) = 2\int_0^\infty \frac{x\sin(ax)}{\left(x^2+b^2\right)^2}\,dx$$

and then differentiate with respect to a to get

$$a\frac{dI(a)}{da} + I(a) = 2\int_0^\infty \frac{x^2\cos(ax)}{\left(x^2+b^2\right)^2}\,dx.$$

The integrand can be re-written in a partial fraction expansion:

$$\frac{x^2\cos(ax)}{\left(x^2+b^2\right)^2} = \frac{\cos(ax)}{x^2+b^2} - \frac{b^2\cos(ax)}{\left(x^2+b^2\right)^2}.$$

Thus,

$$2\int_0^\infty \frac{x^2\cos(ax)}{\left(x^2+b^2\right)^2}\,dx = 2\int_0^\infty \frac{\cos(ax)}{x^2+b^2}dx - 2b^2\int_0^\infty \frac{\cos(ax)}{\left(x^2+b^2\right)^2}\,dx$$

and, since the first integral on right is I(a), we have

$$a\frac{dI(a)}{da} + I(a) = 2I(a) - 2b^2\int_0^\infty \frac{\cos(ax)}{\left(x^2+b^2\right)^2}dx$$

or,

$$a\frac{dI(a)}{da} - I(a) = -2b^2\int_0^\infty \frac{\cos(ax)}{\left(x^2+b^2\right)^2}\,dx.$$

Differentiating this with respect to a, we get

$$a\frac{d^2I(a)}{da^2}+\frac{dI(a)}{da}-\frac{dI(a)}{da}=2b^2\int_0^\infty \frac{x\sin(ax)}{\left(x^2+b^2\right)^2}\,dx$$

or,

$$a\frac{d^2I(a)}{da^2}=2b^2\int_0^\infty \frac{x\sin(ax)}{\left(x^2+b^2\right)^2}\,dx.$$

If you look back at the start of the last paragraph, you'll see that we found the integral on the right to be

$$\int_0^\infty \frac{x\sin(ax)}{\left(x^2+b^2\right)^2}\,dx=\frac{a}{2}\,I(a),$$

and so

$$a\frac{d^2I(a)}{da^2}=2b^2\frac{a}{2}\,I(a)$$

or, rearranging, we have the following *second*-order, linear differential equation for I(a):

$$\frac{d^2I(a)}{da^2}-b^2\,I(a)=0.$$

Such equations are well-known to have exponential solutions—$I(a)=C\ e^{ka}$, where C and k are constants—and substitution into the differential equation gives

$$Ck^2e^{ka}-b^2Ce^{ka}=0$$

and so $k^2-b^2=0$ or $k=\pm b$. Thus, the general solution to the differential equation is the sum of these two particular solutions:

$$I(a)=C_1e^{ab}+C_2e^{-ab}.$$

We need two conditions on I(a) to determine the constants C_1 and C_2, and we can get them from our two different expressions for I(a): the original

$$I(a)=\int_0^\infty \frac{\cos(ax)}{x^2+b^2}\,dx$$

and the expression we got by integrating by parts

$$I(a) = \frac{2}{a}\int_0^{\infty} \frac{x\sin(ax)}{\left(x^2 + b^2\right)^2}\,dx.$$

From the first we see that

$$I(0) = \int_0^{\infty} \frac{1}{x^2 + b^2}\,dx = \frac{1}{b}\left\{\tan^{-1}\left(\frac{x}{b}\right)\right\}\Big|_0^{\infty} = \frac{\pi}{2b},$$

and from the second we see that $\lim_{a\to\infty} I(a) = 0$.

Thus,

$$I(0) = \frac{\pi}{2b} = C_1 + C_2$$

and

$$I(\infty) = 0$$

which says that $C_1 = 0$. Thus, $C_2 = \frac{\pi}{2b}$ and we have this beautiful result:

$$\int_0^{\infty} \frac{\cos(ax)}{x^2 + b^2}\,dx = \frac{\pi}{2b}e^{-ab}, \tag{3.1.7}$$

discovered in 1810 by Laplace. If $b = 1$ and $a = \pi$ then the integral is equal to $\frac{\pi}{2}e^{-\pi} = 0.06788\ \ldots$, and *quad(@(x)cos(pi*x)./(x.^2 + 1),0,1e10)* $= 0.06529\ \ldots$. Before moving on to new tricks, let me observe that with a simple change of variable we can often get some spectacular results from previously derived ones. For example, since (as we showed earlier in (3.1.4))

$$\int_{-\infty}^{\infty} e^{-\frac{x^2}{2}}\,dx = \sqrt{2\pi}$$

it follows (with $u = x\sqrt{2}$) that

$$\int_0^{\infty} e^{-u^2}\,du = \frac{1}{2}\sqrt{\pi}.$$

Then, letting $t = e^{-x^2}$, we have $x = \sqrt{-\ln(t)}$ (and so $dx = -\frac{dt}{2xe^{-x^2}} = -\frac{dt}{2xt}$) and thus