Name: ______________________________

Date: ______________________________

Collaborators: ____

Question:	1	2	3	Total
Points:	20	25	30	75
Score:				

Instructor/grader comments:

Perturbation theory

1. (a) (15 points) Obtain a perturbative solution to

$$\tan(\theta) = \frac{1}{\theta}, \quad \theta \gg 1.$$

Hints/directions:

When θ is large, the graphs of $\tan(\theta)$ and $1/\theta$ intersect near $\theta = N\pi$, where N is a positive integer. Thus, letting

$$\theta = N\pi + x, \quad \tan(\theta) = \tan(x),$$

where x is small, we have

$$\tan x = \frac{1}{N\pi + x}.$$

Now let

$$x = \epsilon a_1 + \epsilon^2 a_2 + \ldots,$$

where

$$\epsilon = \frac{1}{N\pi},$$

and solve for a_1 only. Present your expression for $x(N)$.

Recall that for small s,

$$\tan(s) = s + O(s^3).$$

(b) (5 points) Use a Computer Algebra System to plot in the same figure the graphs of $\tan(\theta)$ and $1/\theta$, as well as your results for intersection points for $0 \le \theta \le 10$.

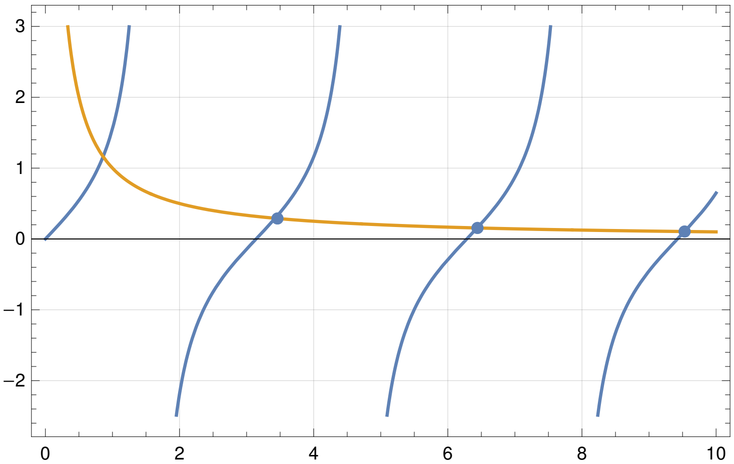
You may use a code similar to the following:

```
T[n_] := n*Pi + your expression here
Show[Plot[{Tan[th],1/th},{th,0,10},GridLines->Automatic,Frame->True],
ListPlot[{{T[1],1/T[1]}, {T[2],1/T[2]}, {T[3],1/T[3]}},
PlotMarkers->Automatic]]
```

Attach a printout of your working CAS session. Be prepared to email your Mathematica notebook file to the grader if requested.

The expected graph is shown in Fig. 1.

Figure 1: Expected result in Problem 1.



2. (a) (20 points) Obtain a perturbative solution to the following initial value problem:

$$\frac{\mathrm{d}^2 \mathrm{y}}{\mathrm{d}x^2} = \sin(x)\,y, \quad y(0) = 1, \quad y'(0) = 1.$$

Hints/directions:

Introduce a small parameter, ϵ, to the problem:

$$\frac{\mathrm{d}^2 \mathrm{y}}{\mathrm{d}x^2} = \epsilon \sin(x)\,y, \quad y(0) = 1, \quad y'(0) = 1.$$

and search for the solution in the form

$$y(x) = y_0(x) + \epsilon\, y_1(x) + \ldots,$$

where

$$y_0(0) = 1, \quad y_0'(0) = 1, \quad y_1(0) = 0, \quad y_1'(0) = 0, \quad \ldots$$

Once you found $y_0(x)$ and $y_1(x)$, take $\epsilon = 1$.

(b) (5 points) Use a Computer Algebra System to plot in the same figure the graphs of numerical solution of the initial value problem, the graph of $y_0(x)$, and the graph of $y(x) = y_0(x) + y_1(x)$ for $0 \le x \le 1$.

You may use a code similar to the following:

```
sol = DSolve[{y''[x]-Sin[x]*y[x]==0,y[0]==1,y'[0]==1},y[x],x]
yexact[x_] := y[x]/.sol[[1]]
y0[x_] := your expression here
```

```
y1[x_] := your expression here
Plot[{yexact[x], y0[x], y0[x] + y1[x]}, {x, 0, 1},
Frame->True,GridLines->Automatic,PlotLegends->"Expressions"]
```

Attach a printout of your working CAS session. Be prepared to email your Mathematica notebook file to the grader if requested.

The expected graph is shown in Fig. 2.

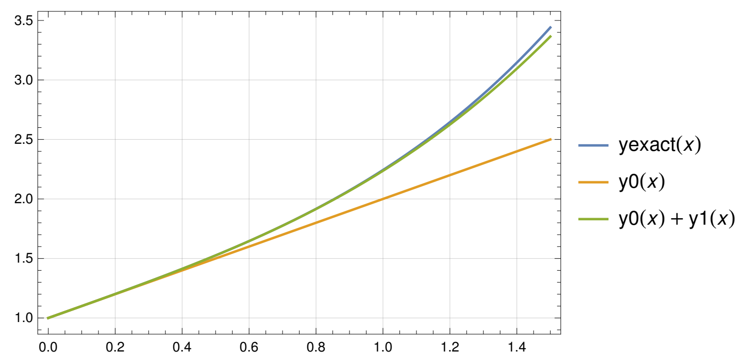


Figure 2: Expected result in Problem 2.

3. (a) (25 points) Find leading-order uniform approximations to the solutions of the following boundary value problem for $0 < \epsilon \ll 1$:

$$\epsilon y'' + \cosh(x) y' - y = 0, \quad y(0) = y(1) = 1, \quad 0 \le x \le 1.$$

Assume that the boundary layer is at $x = 0$.

Recall that

$$\int \frac{dx}{\cosh(x)} = 2\arctan(e^x).$$

(b) (5 points) Use a Computer Algebra System to plot in the same figure the graphs of numerical solution of the initial value problem, the graph of the inner solution, the graph of the outer solution, and the graph of the uniform approximation for $\epsilon = 1/10$ and $0 \le x \le 1$.

You may use a code similar to the following:

```
eps = 1/10
eq = eps*y''[x] + Cosh[x]*y'[x] - y[x] == 0
sol = NDSolve[{eq, y[0] == 1, y[1] == 1}, y, x]
yode = y[x] /. sol
ynumer[x_] := yode
youter[x_] := your code here
```

```
yinner[x_] := your code here
yuniform[x_] := your code here
Plot[{ynumer[x],youter[x],yinner[x],yuniform[x]},{x,0,1},Frame->True,
GridLines -> Automatic, PlotLegends->"Expressions"]
```

Attach a printout of your working CAS session. Be prepared to email your Mathematica notebook file to the grader if requested.

The expected graph is shown in Fig. 3.

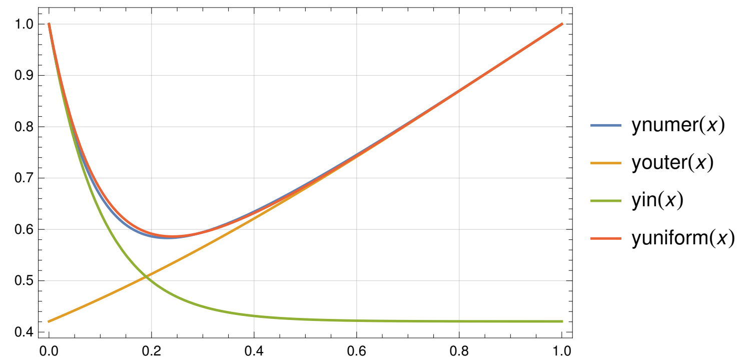


Figure 3: Expected result in Problem 3.