Name: ______________________________

Date: ______________________________

Collaborators: ____

Show all your work and indicate your reasoning in order to receive the credit.

Question:	1	2	3	Total
Points:	25	30	15	70
Score:				

Instructor/grader comments:

Leibniz' rule

1. (25 points) Find the positive value of x that maximizes the value of the following integral

$$I(x) = \int_{x^2-1}^{x^2+1} \frac{\mathrm{d}u}{\Gamma(u)}, \qquad (B)$$

where $\Gamma(u)$ is the Gamma function.

For reference, the integrand in Eq. (B) is sketched in Figure 1.

Present your solution as an **algebraic expression**, i.e. as an expression built up from constants, variables, and the algebraic operations (addition, subtraction, multiplication, division and exponentiation by an exponent that is a rational number), as well as a decimal number with three significant digits.

To verify that your answer makes sense, draw the vertical line $u = x^2$ where is x your solution in the figure 1.

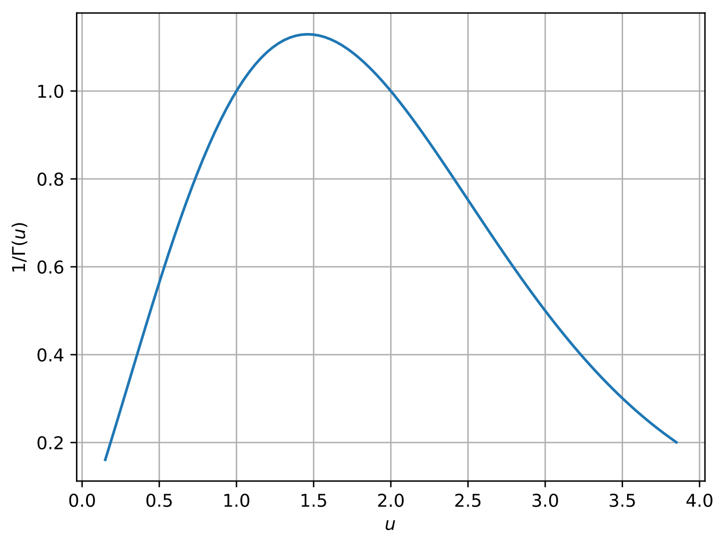

Figure 1: Graph of $1/\Gamma(u)$.

Hints: Recall that the derivative of a function at its maximum is zero. Find the derivative of $I(x)$. Simplify the equation that you obtained using the relations $\Gamma(x+1) = x\Gamma(x)$, $\Gamma(x) = (x-1)\Gamma(x-1)$. Solve the equation and select the correct root.

2. (30 points) Find the solution of the following integral equation:

$$f(x) = \int_0^\infty e^{-|x-s|} f(s)\, \mathrm{d}s, \quad 0 \le x < \infty. \qquad (C)$$

Equation (C) is a linear integral equation, therefore its solution can be determined up to an arbitrary multiplication constant. Fix that constant by requiring that

$$f(0) = 1. \qquad (D)$$

Note that combining (C) and (D) we get an additional relation $\int_0^\infty e^{-s} f(s)\, \mathrm{d}s = 1$.

The recommended steps to the solution are as follows:

a. Rewrite the integral in the right hand side of Eq. (C) as the sum of two integrals – one for $0 \le s \le x$ and another for $x \le s < \infty$. In each of the integrals replace $|x - s|$ with an expression that does not use the absolute values.

b. Differentiate the "new" form of the integral equation. Simplify the expression. Use it to find the value of $f'(0)$.

c. Differentiate the first derivative of the integral equation once more. Simplify the second derivative of the equation to reduce it to the second order ordinary differential equation.

   Hint: you should get the familiar equation for harmonic oscillations.

d. Write down the general solution of the differential equation. Determine the integration constants using $f(0)$ and $f'(0)$ that you already found.

e. Use Mathematica to verify your solution by substituting it back to the integral equation (in its "new" form).

   Hint: the Mathematica code could be something like the following:

```
eq = f[x] - Integrate[your_first_integrand, {s, 0, x}] -
            Integrate[your_second_integrand, {s, x, Infinity}]
f[x_] := your_solution
eq
```

   Print your Mathematica session (use "File" → "Save as" menu) and attach the printout to the rest of your homework.

Gamma and Beta functions

3. Evaluate the following expressions. Here Γ and B are Euler gamma and beta functions. Only the values of $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$ and $\Gamma(1) = 1$ are known.

   (a) (3 points) $\Gamma\left(\frac{5}{2}\right)$

   (b) (6 points) $\Gamma\left(-\frac{3}{2}\right)$

   (c) (3 points) $\Gamma(5)$

   (d) (3 points) $B\left(\frac{1}{2}, \frac{5}{2}\right)$