

EVALUATE GAUSSIAN AND FRESNEL'S INTEGRALS USING DIFFERENTIATION UNDER THE INTEGRAL SIGN

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https://www.phys.uconn.edu/~rozman/Courses/P2400_24S/

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In this note we evaluate Gaussian integral,

$$I_g \equiv \int_0^{\infty} e^{-x^2} dx = \frac{\sqrt{\pi}}{2}, \quad (1)$$

and Fresnel integrals,

$$I_c \equiv \int_0^{\infty} \cos(x^2) dx = \frac{\sqrt{\pi}}{2\sqrt{2}} \quad (2)$$

and

$$I_s \equiv \int_0^{\infty} \sin(x^2) dx = \frac{\sqrt{\pi}}{2\sqrt{2}}, \quad (3)$$

using differentiation under the integral sign.

The two parts of this note can be read independently.

Gaussian integral

Consider the following integral

$$J(x) = \int_0^{\infty} \frac{e^{-x^2(1+y^2)}}{1+y^2} dy. \quad (4)$$

We know its values for $x = 0$:

$$J(0) = \int_0^{\infty} \frac{dy}{1+y^2} = \arctan(\infty) = \frac{\pi}{2}, \quad (5)$$

and for $x = \infty$:

$$J(\infty) = 0. \quad (6)$$

The derivative of $J(x)$,

$$\begin{aligned} \frac{dJ}{dx} &= -2x \int_0^{\infty} e^{-x^2(1+y^2)} dy = -2e^{-x^2} \int_0^{\infty} e^{-(xy)^2} d(xy) \\ &= -2e^{-x^2} \int_0^{\infty} e^{-u^2} du \\ &= -2e^{-x^2} I_g. \end{aligned} \quad (7)$$

Integrating Eq. (7) with respect to x from 0 to ∞ , we obtain:

$$J(\infty) - J(0) = -2I_g \int_0^{\infty} e^{-x^2} dx = -2I_g^2, \quad (8)$$

or, using Eq. (5) and Eq. (6),

$$I_g^2 = \frac{J(0)}{2} = \frac{\pi}{4}, \quad (9)$$

i.e.

$$I_g = \frac{\sqrt{\pi}}{2}. \quad (10)$$

Fresnel's integrals

Consider the following integrals:

$$c(x) = \int_0^{\infty} \frac{\cos(x^2(1+y^2))}{1+y^2} dy, \quad (11)$$

$$s(x) = \int_0^{\infty} \frac{\sin(x^2(1+y^2))}{1+y^2} dy. \quad (12)$$

We know their values for $x = 0$:

$$c(0) = \int_0^{\infty} \frac{dy}{1+y^2} = \arctan(\infty) = \frac{\pi}{2}, \quad (13)$$

$$s(0) = 0, \quad (14)$$

and for $x = \infty$:

$$c(\infty) = s(\infty) = 0. \quad (15)$$

The derivative of $c(x)$,

$$\begin{aligned} \frac{dc}{dx} &= -2x \int_0^{\infty} \sin(x^2(1+y^2)) dy = -2x \int_0^{\infty} \sin(x^2 + (xy)^2) dy \\ &= -2 \sin(x^2) \int_0^{\infty} \cos(xy)^2 d(xy) - 2 \cos(x^2) \int_0^{\infty} \sin(xy)^2 d(xy) \\ &= -2 \sin(x^2) \int_0^{\infty} \cos(u)^2 du - 2 \cos(x^2) \int_0^{\infty} \sin(u)^2 du \\ &= -2 \sin(x^2) I_c - 2 \cos(x^2) I_s. \end{aligned} \quad (16)$$

Similarly,

$$\frac{ds}{dx} = 2 \cos(x^2) I_c - 2 \sin(x^2) I_s. \quad (17)$$

Integrating Eqs. (16), (17) with respect to x from 0 to ∞ , we obtain:

$$c(\infty) - c(0) = -2 I_c \int_0^{\infty} \sin(x^2) dx - 2 I_s \int_0^{\infty} \cos(x^2) dx = -4 I_c I_s, \quad (18)$$

$$s(\infty) - s(0) = 2 I_c \int_0^{\infty} \cos(x^2) dx - 2 I_s \int_0^{\infty} \sin(x^2) dx = 2 I_c^2 - 2 I_s^2. \quad (19)$$

Using the boundary conditions Eqs. (13)-(15), $c(\infty) - c(0) = -\frac{\pi}{2}$, $s(\infty) - s(0) = 0$, therefore

$$I_c I_s = \frac{\pi}{8}, \quad I_c = I_s. \quad (20)$$

Finally,

$$I_c = I_s = \frac{\sqrt{\pi}}{2\sqrt{2}}. \quad (21)$$

References

- [1] J. van Yzeren, “Moivre’s and Fresnel’s integrals by simple integration,” *The American Mathematical Monthly*, vol. 86, no. 8, pp. 690–693, 1979.