EULER'S INTEGRALS

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https://www.phys.uconn.edu/~rozman/Courses/P2400_24S/

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The Gamma function

In the eighteenth century, Leonhard Euler (1707-1783) concerned himself with the problem of interpolating between the numbers $n! \equiv n \cdot (n-1) \cdot (n-2) \cdot \ldots \cdot 2 \cdot 1$. This problem led Euler in 1729 to the now famous *gamma function*, a generalization of the factorial function that gives meaning to $x!$ when x is any positive number. Furthermore, the factorial function can be extended to certain negative numbers and as well as to complex numbers.

Consider the integral

$$\Gamma(x) \equiv \int_0^\infty t^{x-1} e^{-t}\, \mathrm{d}t \tag{1}$$

which is convergent for $x - 1 > -1$, i.e. $x > 0$. Using the relation $e^{-t}\, \mathrm{d}t = -\mathrm{d}\left(e^{-t}\right)$ and integrating Eq. (1) by parts once, we get:

$$\begin{aligned}\Gamma(x) &= -\int_0^\infty t^{x-1}\, \mathrm{d}\left(e^{-t}\right) = \underbrace{-\, t^{x-1} e^{-t}\Big|_0^\infty}_{\to 0} + \int_0^\infty e^{-t}\, \mathrm{d}\left(t^{x-1}\right) \\ &= (x-1) \int_0^\infty t^{x-2} e^{-t}\, \mathrm{d}t = (x-1)\,\Gamma(x-1),\end{aligned} \tag{2}$$

where in order the integral for $\Gamma(x-1)$ to exist, x must be large than 1.

Repeatedly integrating by parts, we obtain,

$$\Gamma(x) = (x-1)\,\Gamma(x-1) = (x-1)(x-2)\,\Gamma(x-2) = (x-1)(x-2)(x-3)\,\Gamma(x-3) = \ldots \tag{3}$$

If x = n is a positive integer, then repeating the integration by parts we eventually arrive to the expression:

$$\Gamma(n) = (n-1)\cdot(n-2)\cdot\ldots\cdot 2\cdot\Gamma(1). \tag{4}$$

Noticing that

$$\Gamma(1) = \int_0^\infty e^{-t}\,\mathrm{d}t = 1, \tag{5}$$

we conclude that for an positive integer argument n,

$$\Gamma(n) = (n-1)! \tag{6}$$

The gamma function has been used as a means of generalizing certain functions, operations, etc., that are commonly defined in terms of factorials. In addition, the gamma function is useful in the evaluation of many non-elementary integrals and in the definition of other special functions.

To obtain another useful integral representation of $\Gamma(x)$, let's set the integration variable in Eq. (1) to $t = u^2$, $\mathrm{d}t = 2u\mathrm{d}u$, $0 \le u < \infty$. We get

$$\Gamma(x) = 2\int_0^\infty u^{2x-1}e^{-u^2}\,\mathrm{d}u. \tag{7}$$

Example 1. Evaluate the integral:

$$I = \int_0^\infty x^4 e^{-x^3}\mathrm{d}x.$$

Let $u = x^3$, then

$$x = u^{\frac{1}{3}}, \quad x^4 = u^{\frac{4}{3}}, \quad \mathrm{d}x = \frac{1}{3}u^{-\frac{2}{3}}\mathrm{d}u, \quad 0 \le u < \infty.$$

$$I = \int_0^\infty x^4 e^{-x^3}\mathrm{d}x = \frac{1}{3}\int_0^\infty u^{\frac{4}{3}}u^{-\frac{2}{3}}e^{-u}\mathrm{d}u = \frac{1}{3}\int_0^\infty u^{\frac{2}{3}}e^{-u}\mathrm{d}u = \frac{1}{3}\int_0^\infty u^{\frac{5}{3}-1}e^{-u}\mathrm{d}u = \frac{1}{3}\Gamma\left(\frac{5}{3}\right).$$

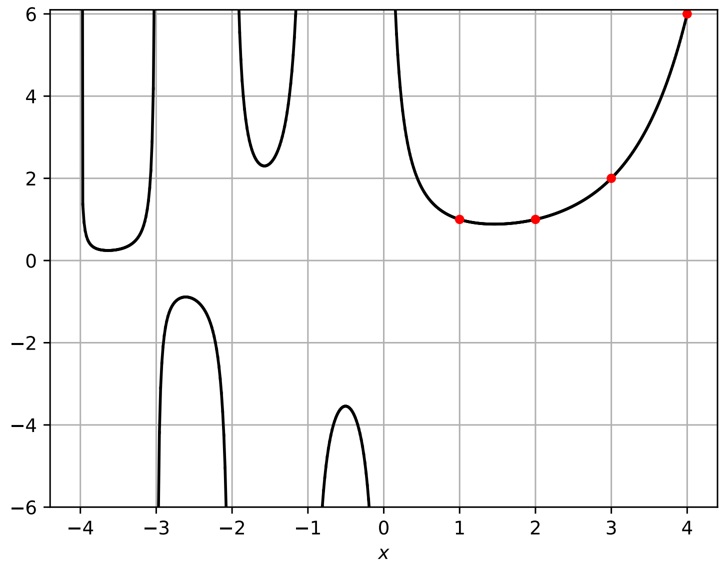

Figure 1: The gamma function. Red dots mark the values of n!, n = 1 – 4.

Fractional-order derivatives

In addition to generalizing the notion of factorials, the gamma function can be used in a variety of situations to transform a discrete processes into a continuous one.

We can illustrate the concept of fractional derivatives by first recalling the derivative formula from calculus:

$$\frac{\mathrm{d}^n}{\mathrm{d}x^n}\, x^a = a\,(a-1)\cdots(a-n+1)\,x^{a-n}, \quad a \geq 0, \quad n = 1, 2, 3, \ldots \tag{8}$$

In terms of the gamma function, we can rewrite Eq. (8) as follows:

$$\frac{\mathrm{d}^n}{\mathrm{d}x^n}\, x^a = \frac{\Gamma(a+1)}{\Gamma(a-n+1)}\, x^{a-n}. \tag{9}$$

The right-hand side of this expression is meaningful for any real number n for which $\Gamma(a-n+1)$ is defined. Hence, we assume that the same is true of the left-hand side and write

$$\frac{\mathrm{d}^\nu}{\mathrm{d}x^\nu}\, x^a = \frac{\Gamma(a+1)}{\Gamma(a-\nu+1)}\, x^{a-\nu}, \tag{10}$$

where ν is not restricted to integer values. Equation Eq. (10)) provides a method of computing *fractional-order derivatives* of polynomials.

1 The Beta function

Another function useful in various applications is the related to gamma function beta function, often called the *eulerian integral of the first kind*.

$$B(x,y) \equiv \int_0^1 t^{x-1}(1-t)^{y-1}\mathrm{d}t. \tag{11}$$

If we make the change of variable $u = 1 - t$, we find

$$B(x,y) = \int_0^1 (1-u)^{x-1} u^{y-1}\mathrm{d}t. \tag{12}$$

from which we deduce the symmetry property

$$B(x,y) = B(y,x). \tag{13}$$

We obtain another representation of the Beta function if we make the following change of integration variable:

$$t = \sin^2\theta, \quad 0 \le \theta \le \frac{\pi}{2}, \quad \mathrm{d}t = 2\cos\theta\,\sin\theta\,\mathrm{d}\theta, \tag{14}$$

$$B(x,y) = 2\int_0^{\frac{\pi}{2}} \sin^{2x-1}\theta\,\cos^{2y-1}\theta\,\mathrm{d}\theta. \tag{15}$$

To establish the relation between Beta and Gamma functions, let's calculate the following product using the integral representation Eq. (7) for Gamma function:

$$\Gamma(x)\Gamma(y) = 4\int_0^\infty u^{2x-1}e^{-u^2}\,\mathrm{d}u \int_0^\infty v^{2y-1}e^{-v^2}\,\mathrm{d}v = 4\iint_0^\infty u^{2x-1}v^{2y-1}e^{-(u^2+v^2)}\,\mathrm{d}u\mathrm{d}v. \tag{16}$$

The presence of the term $u^2 + v^2$ in the integrand suggests the change of variables from cartesian (u,v) to polar (r,θ):

$$u = r\cos\theta, \quad v = r\sin\theta, \quad r^2 = u^2 + v^2, \quad 0 \le r < \infty, \quad 0 \le \theta \le \frac{\pi}{2}, \quad \mathrm{d}u\mathrm{d}v \to r\,\mathrm{d}r\mathrm{d}\theta. \tag{17}$$

Thus,

$$\Gamma(x)\,\Gamma(y) = 4\int_0^\infty r^{2(x+y)-1} e^{-r^2}\,\mathrm{d}r \int_0^{\frac{\pi}{2}} \sin^{2x-1}\theta\,\cos^{2y-1}\theta\,\mathrm{d}\theta = \Gamma(x+y)\,B(x,y). \tag{18}$$

Therefore,

$$B(x,y) = \frac{\Gamma(x)\,\Gamma(y)}{\Gamma(x+y)}. \tag{19}$$

We now use Eq. (19) to calculate the value $\Gamma\left(\frac{1}{2}\right)$. On the one hand, from Eq. (15), $B\left(\frac{1}{2},\frac{1}{2}\right) = \pi$. On the other hand, from Eq. (19), $B\left(\frac{1}{2},\frac{1}{2}\right) = \Gamma^2\left(\frac{1}{2}\right)$. Therefore,

$$\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}. \tag{20}$$

Example 1. Find the area enclosed by the curve $x^4 + y^4 = 1$ (see Fig. 2).

$$y(x) = \left(1 - x^4\right)^{\frac{1}{4}}, \quad -1 \le x \le 1.$$

$$A = 4\int_0^1 y(x)\,\mathrm{d}x.$$

Let $u = x^4$, then

$$x = u^{\frac{1}{4}}, \quad \mathrm{d}x = \frac{1}{4}u^{-\frac{3}{4}}\mathrm{d}u, \quad 0 \le u \le 1.$$

$$A = 4\,\frac{1}{4}\int_0^1 u^{-\frac{3}{4}}(1-u)^{\frac{1}{4}}\,\mathrm{d}u = \int_0^1 u^{\frac{1}{4}-1}(1-u)^{\frac{5}{4}-1}\,\mathrm{d}u = B\left(\frac{1}{4},\frac{5}{4}\right) = \frac{\Gamma\left(\frac{1}{4}\right)\Gamma\left(\frac{5}{4}\right)}{\Gamma\left(\frac{3}{2}\right)}.$$

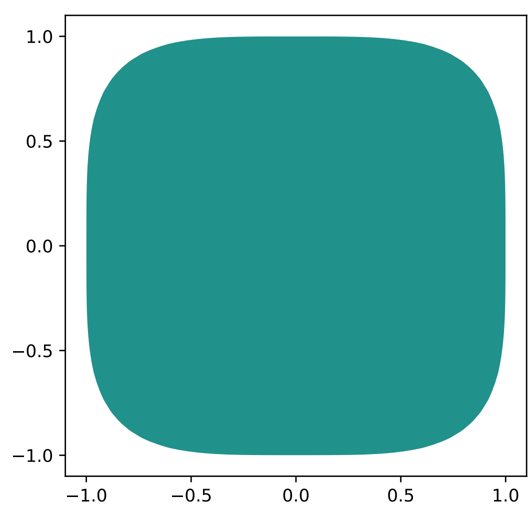

Figure 2: The area enclosed by the curve $x^4 + y^4 = 1$