DUPLICATION FORMULA FOR GAMMA FUNCTION

Spring semester 2024

https://www.phys.uconn.edu/~rozman/Courses/P2400_24S/

Last modified: January 19, 2024

Gamma function satisfies the following identity for all complex z:

$$\Gamma(2z) = \frac{2^{2z-1}}{\sqrt{\pi}}\Gamma(z)\Gamma\left(z+\frac{1}{2}\right), \tag{1}$$

referred to as Legendre *duplication formula*.

We start from the integral expression of Beta function of equal arguments:

$$B(z,z) = \int_0^1 x^{z-1}(1-x)^{z-1}\mathrm{d}x. \tag{2}$$

Perform the substitution $x = \frac{1+t}{2}$, so that $-1 \le t \le 1$ and $\mathrm{d}x = \frac{1}{2}\mathrm{d}t$. This transforms Eq. (2) into

$$B(z,z) = 2^{2-2z}\frac{1}{2}\int_{-1}^1 (1-t)^{z-1}(1+t)^{z-1}\mathrm{d}t = 2^{2-2z}\int_0^1 (1-t^2)^{z-1}\mathrm{d}t. \tag{3}$$

Changing the integration variable in the last integral to $u = t^2$, so that $t = u^{\frac{1}{2}}$, $0 \le u \le 1$, and $\mathrm{d}t = \frac{1}{2}u^{-\frac{1}{2}}$, we transform the integral in Eq. (3) to

$$B(z,z) = 2^{1-2z}\int_0^1 u^{-\frac{1}{2}}(1-u)^{z-1}\mathrm{d}u = 2^{1-2z}B\left(\frac{1}{2},z\right). \tag{4}$$

Thus we have the following identity:

$$B(z,z) = 2^{1-2z} B\left(\frac{1}{2}, z\right). \tag{5}$$

In terms of Gamma functions,

$$B(z,z) = \frac{\Gamma(z)\Gamma(z)}{\Gamma(2z)}, \tag{6}$$

$$B\left(\frac{1}{2}, z\right) = \frac{\Gamma\left(\frac{1}{2}\right)\Gamma(z)}{\Gamma\left(z+\frac{1}{2}\right)}, \tag{7}$$

so that Eq. (5) is:

$$\frac{\Gamma(z)\Gamma(z)}{\Gamma(2z)} = 2^{1-2z} \frac{\Gamma\left(\frac{1}{2}\right)\Gamma(z)}{\Gamma\left(z+\frac{1}{2}\right)}. \tag{8}$$

Cancelling the common factors $\Gamma(z)$ in both sides of Eq. (8), rearranging terms, and using the value $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$, we see that

$$\Gamma(2z) = \frac{2^{2z-1}}{\sqrt{\pi}} \Gamma(z)\Gamma\left(z+\frac{1}{2}\right). \tag{9}$$