1 The case of the integrand depending on a parameter

Let

\[ I(t) = \int_{a}^{b} f(x,t) \, dx, \tag{1} \]

where \( a, b \) are fixed parameters.

Then,

\[ \frac{dI}{dt} = \frac{d}{dt} \left( \int_{a}^{b} f(x,t) \, dx \right) = \int_{a}^{b} \left( \frac{\partial}{\partial t} f(x,t) \right) \, dx, \tag{2} \]

Indeed, considering the definition of the derivative as the limit,

\[ \frac{dI}{dt} = \lim_{\Delta t \to 0} \frac{I(t + \Delta t) - I(t)}{\Delta t}, \tag{3} \]

and expanding \( f(x,t + \Delta t) \) into Taylor series,

\[ f(x,t + \Delta t) = f(x,t) + \frac{\partial f}{\partial t} \Delta t + O(\Delta t^2). \tag{4} \]
\[ I(t + \Delta t) - I(t) = \int_{a}^{b} f(x, t + \Delta t) \, dx - \int_{a}^{b} f(x, t) \, dx = \int_{a}^{b} \left[ f(x, t + \Delta t) - f(x, t) \right] \, dx \]
\[ = \int_{a}^{b} \left[ \frac{\partial f}{\partial t} \Delta t + O(\Delta t^2) \right] \, dx = \int_{a}^{b} \left( \frac{\partial f}{\partial t} \right) \, dx \Delta t + O(\Delta t^2). \]  

Substituting Eq. (5) into Eq. (3), and taking the limit \( \Delta t \to 0 \), we obtain Eq. (2).

Figure 1: \( I(t) \) (gray background), \( I(t + \Delta t) \) (hatched background), and their difference in Eq. (5).

2 Case of the integration range depending on a parameter

Let
\[ I(t) = \int_{a(t)}^{b(t)} f(x) \, dx. \]  

where the integration limits \( a(t) \) and \( b(t) \) are functions of the parameter \( t \) but the integrand \( f(x) \) does not depend on \( t \).
Then,
\[
\frac{dI}{dt} = \frac{d}{dt} \left( \int_{a(t)}^{b(t)} f(x) \, dx \right) = f(b(t)) \frac{db}{dt} - f(a(t)) \frac{da}{dt},
\]  
(7)

Indeed,
\[
\frac{dI}{dt} = \lim_{\Delta t \to 0} \frac{I(t + \Delta t) - I(t)}{\Delta t}
\]  
(8)

\[
I(t + \Delta t) = \int_{a(t+\Delta t)}^{b(t+\Delta t)} f(x) \, dx.
\]  
(9)

\[
I(t + \Delta t) - I(t) = \int_{a(t+\Delta t)}^{b(t+\Delta t)} f(x) \, dx - \int_{a(t)}^{b(t+\Delta t)} f(x) \, dx = \int_{a(t)}^{b(t)} f(x) \, dx - \int_{a(t+\Delta t)}^{b(t)} f(x) \, dx
\]
\[
= \left( b(t + \Delta t) - b(t) \right) f(b(t)) - \left( a(t + \Delta t) - a(t) \right) f(a(t)) + O(\Delta t^2)
\]
\[
= \left[ f(b(t)) \frac{db}{dt} - f(a(t)) \frac{da}{dt} \right] \Delta t + O(\Delta t^2),
\]  
(10)

where we used that \( a(t + \Delta t) - a(t) = \frac{da}{dt} \Delta t + O(\Delta t^2) \) and similar for \( b \).

Combining Eq. (10) and Eq. (8), and taking the limit \( \Delta t \to 0 \), we obtain Eq. (7).

3 General case
\[
\frac{d}{dt} \left( \int_{a(t)}^{b(t)} f(x,t) \, dx \right) = \int_{a(t)}^{b(t)} \left( \frac{\partial}{\partial t} f(x,t) \right) \, dx + f(b(t)) \frac{db}{dt} - f(a(t)) \frac{da}{dt}
\]  
(11)
Figure 2: \( I(t) \) (gray background), \( I(t+\Delta t) \) (hatched background), and their difference in Eq. (10).

Figure 3: \( I(t) \) (gray background), \( I(t+\Delta t) \) (hatched background), and their difference.
4 Examples

**Problem 1.** Find the solution of the following integral equation:

\[
\phi(x) + \frac{1}{2} \int_{-1}^{1} |x-s| \phi(s) \, ds = x, \quad -1 \leq x \leq 1. \tag{12}
\]

**Solution:** we rewrite the integral term in the equation as follows,

\[
\hat{L} \phi(x) \equiv \int_{-1}^{1} |x-s| \phi(s) \, ds = \int_{-1}^{x} (x-s) \phi(s) \, ds + \int_{x}^{1} (s-x) \phi(s) \, ds, \quad -1 \leq x \leq 1, \tag{13}
\]

where in the first integral \(x \geq s\) and \(|x-s| = x-s\); in the second integral \(x \leq s\) and \(|x-s| = s-x\). The integral equation get the form:

\[
\phi(x) + \frac{1}{2} \int_{-1}^{x} (x-s) \phi(s) \, ds + \frac{1}{2} \int_{x}^{1} (s-x) \phi(s) \, ds = x. \tag{14}
\]

Taking the derivative of Eq. (14) with respect to \(x\) and using the result Eq. (11), we obtain:

\[
\phi'(x) + \frac{1}{2} \int_{-1}^{x} \phi(s) \, ds - \frac{1}{2} \int_{x}^{1} \phi(s) \, ds = 1. \tag{15}
\]

Taking the derivative of Eq. (15), we obtain the following ordinary differential equation:

\[
\phi''(x) + \phi(x) = 0. \tag{16}
\]

The general solution of Eq. (16) is as follows:

\[
\phi(x) = A \cos(x) + B \sin(x), \tag{17}
\]

where \(A\) and \(B\) are the integration constants. To find them we plug the solution Eq. (17) back into the integral equation and set \(x = 0\):

\[
\phi(0) = A, \tag{18}
\]

\[
\hat{L} \phi(0) = \frac{1}{2} \int_{-1}^{1} |0-s| \phi(s) \, ds = \frac{A}{2} \int_{-1}^{1} |s| \cos(s) \, ds + \frac{B}{2} \int_{-1}^{1} |s| \sin(s) \, ds = \alpha A, \tag{19}
\]
where

\[ \alpha \equiv \int_{0}^{1} s \cos(s) ds. \]  

(20)

\( \alpha \) is positive, since the integrand is positive. (Also, \( \alpha = \sin(1) + \cos(1) - 1 \) but we do not need the exact value.)

The equation

\[ A + \alpha A = 0, \]

(21)

where \( \alpha > 0 \) has the solution

\[ A = 0. \]

(22)

Thus,

\[ \phi(x) = B \sin(x). \]

(23)

To determine the constant \( B \) we plug the solution Eq. (23) into Eq. (15) and set \( x = 1 \):

\[ \phi'(x) = B \cos(x), \quad \phi'(1) = B \cos(1), \]

(24)

\[ \frac{1}{2} \int_{-1}^{1} \phi(s) ds = \frac{B}{2} \int_{-1}^{1} \sin(s) ds = 0, \]

(25)

\[ B \cos(1) = 1, \]

(26)

\[ B = \frac{1}{\cos(1)}. \]

(27)

Therefore, the solution of Eq. (12) is as follows:

\[ \phi(x) = \frac{\sin x}{\cos(1)}, \quad -1 \leq x \leq 1. \]

(28)