DIFFERENTIATING UNDER THE INTEGRAL SIGN

Spring semester 2024

https://www.phys.uconn.edu/~rozman/Courses/P2400_24S/

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1 The case of the integrand depending on a parameter

Let

$$I(t) = \int_a^b f(x,t)\,\mathrm{d}x, \tag{1}$$

where a, b are fixed parameters.

Then,

$$\frac{\mathrm{d}I}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t}\left(\int_a^b f(x,t)\,\mathrm{d}x\right) = \int_a^b \left(\frac{\partial}{\partial t} f(x,t)\right)\mathrm{d}x, \tag{2}$$

Indeed, considering the definition of the derivative as the limit,

$$\frac{\mathrm{d}I}{\mathrm{d}t} = \lim_{\Delta t \to 0} \frac{I(t+\Delta t) - I(t)}{\Delta t}, \tag{3}$$

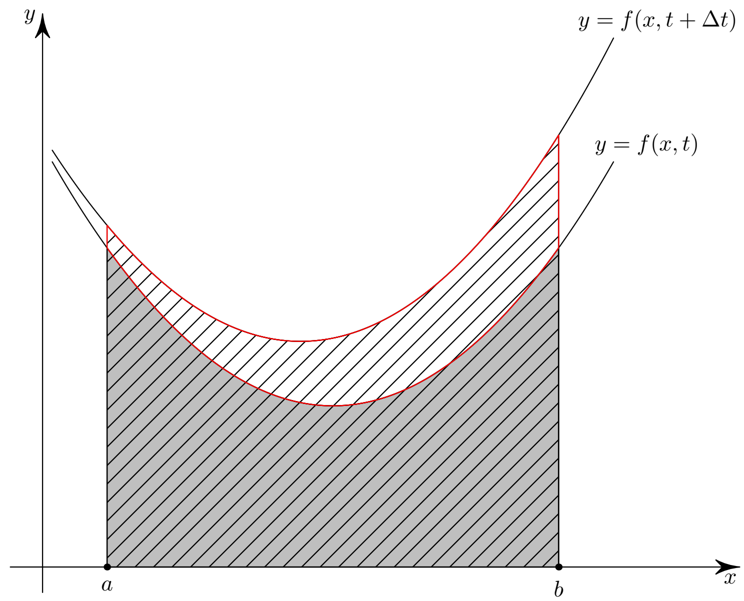
and expanding $f(x, t+\Delta t)$ into Taylor series,

$$f(x,t+\Delta t) = f(x,t) + \frac{\partial f}{\partial t}\Delta t + O\left(\Delta t^2\right). \tag{4}$$

$$I(t+\Delta t)-I(t) = \int_a^b f(x,t+\Delta t)\,\mathrm{d}x - \int_a^b f(x,t)\,\mathrm{d}x = \int_a^b \Big[f(x,t+\Delta t)-f(x,t)\Big]\,\mathrm{d}x$$
$$= \int_a^b \left[\frac{\partial f}{\partial t}\Delta t + O\big(\Delta t^2\big)\right]\mathrm{d}x = \left[\int_a^b \left(\frac{\partial f}{\partial t}\right)\mathrm{d}x\right]\Delta t + O\big(\Delta t^2\big). \tag{5}$$

Substituting Eq. (5) into Eq. (3), and taking the limit $\Delta t \to 0$, we obtain Eq. (2).

Figure 1: $I(t)$ (gray background), $I(t+\Delta t)$ (hatched background), and their difference in Eq. (5).



2 Case of the integration range depending on a parameter

Let

$$I(t) = \int_{a(t)}^{b(t)} f(x)\,\mathrm{d}x. \tag{6}$$

where the integration limits $a(t)$ and $b(t)$ are functions of the parameter t but the integrand $f(x)$ does not depend on t.

Then,

$$\frac{\mathrm{dI}}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t}\left(\int\limits_{a(t)}^{b(t)} f(x)\,\mathrm{d}x\right) = f\left(b(t)\right)\frac{\mathrm{d}b}{\mathrm{d}t} - f\left(a(t)\right)\frac{\mathrm{d}a}{\mathrm{d}t}, \tag{7}$$

Indeed,

$$\frac{\mathrm{d}I}{\mathrm{d}t} = \lim_{\Delta t\to 0}\frac{I(t+\Delta t)-I(t)}{\Delta t} \tag{8}$$

$$I(t+\Delta t) = \int\limits_{a(t+\Delta t)}^{b(t+\Delta t)} f(x)\,\mathrm{d}x. \tag{9}$$

$$\begin{aligned} I(t+\Delta t)-I(t) &= \int\limits_{a(t+\Delta t)}^{b(t+\Delta t)} f(x)\,\mathrm{d}x - \int\limits_{a(t)}^{b(t)} f(x)\,\mathrm{d}x = \int\limits_{b(t)}^{b(t+\Delta t)} f(x)\,\mathrm{d}x - \int\limits_{a(t)}^{a(t+\Delta t)} f(x)\,\mathrm{d}x \\ &= \left(b(t+\Delta t)-b(t)\right) f\left(b(t)\right) - \left(a(t+\Delta t)-a(t)\right) f\left(a(t)\right) + O\left(\Delta t^2\right) \\ &= \left[f\left(b(t)\right)\frac{\mathrm{d}b}{\mathrm{d}t} - f\left(a(t)\right)\frac{\mathrm{d}a}{\mathrm{d}t}\right]\Delta t + O\left(\Delta t^2\right), \end{aligned} \tag{10}$$

where we used that $a(t+\Delta t)-a(t) = \frac{\mathrm{d}a}{\mathrm{d}t}\Delta t + O\left(\Delta t^2\right)$ and similar for b.

Combining Eq. (10) and Eq. (8), and taking the limit $\Delta t \to 0$, we obtain Eq. (7).

3 General case

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\int\limits_{a(t)}^{b(t)} f(x,t)\,\mathrm{d}x\right) = \int\limits_{a(t)}^{b(t)}\left(\frac{\partial}{\partial t} f(x,t)\right)\mathrm{d}x + f\left(b(t)\right)\frac{\mathrm{d}b}{\mathrm{d}t} - f\left(a(t)\right)\frac{\mathrm{d}a}{\mathrm{d}t} \tag{11}$$

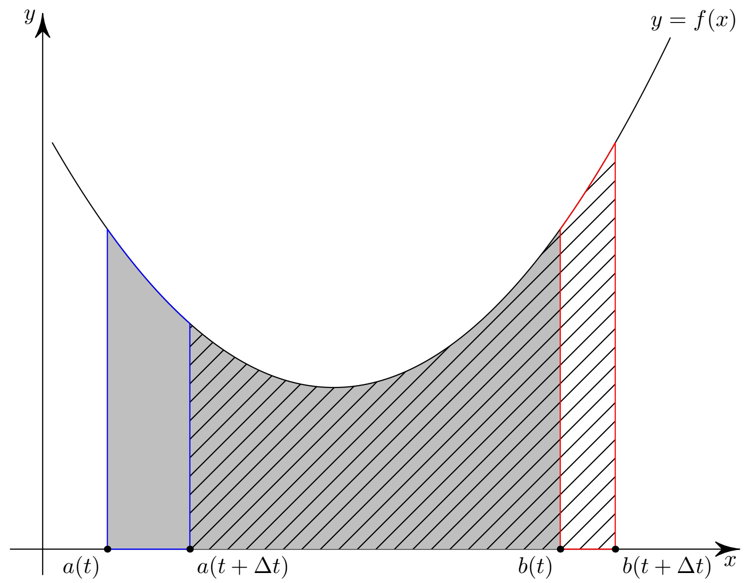


Figure 2: $I(t)$ (gray background), $I(t+\Delta t)$ (hatched background), and their difference in Eq. (10).

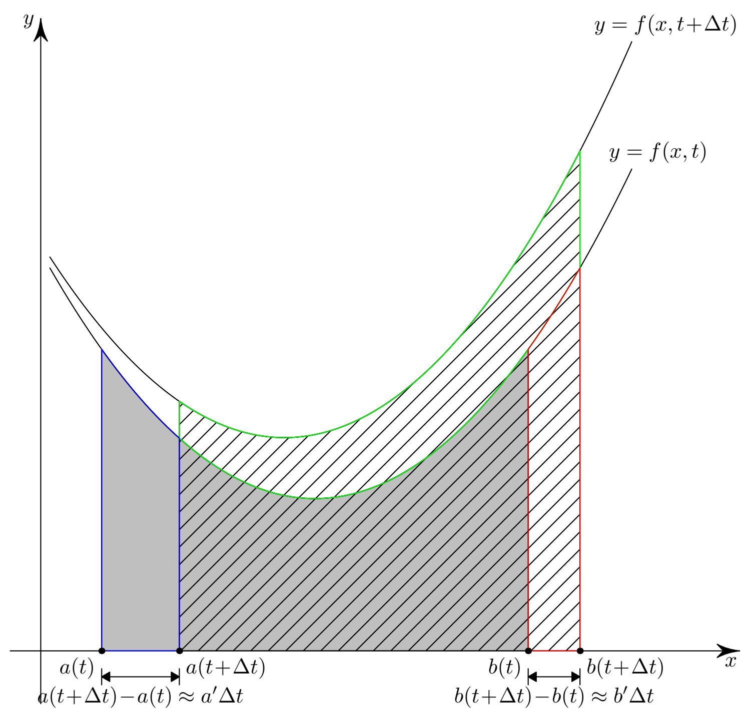


Figure 3: $I(t)$ (gray background), $I(t+\Delta t)$ (hatched background), and their difference.

4 Examples

Problem 1. Find the solution of the following integral equation:

$$\phi(x)+\frac{1}{2}\int_{-1}^{1}|x-s|\,\phi(s)\,\mathrm{d}s = x, \quad -1\le x\le 1. \tag{12}$$

Solution: we rewrite the integral term in the equation as follows,

$$\left(\hat{L}\phi\right)(x)\equiv\int_{-1}^{1}|x-s|\,\phi(s)\,\mathrm{d}s=\int_{-1}^{x}(x-s)\,\phi(s)\,\mathrm{d}s+\int_{x}^{1}(s-x)\,\phi(s)\,\mathrm{d}s, \quad -1\le x\le 1, \tag{13}$$

where in the first integral $x\ge s$ and $|x-s|=x-s$; in the second integral $x\le s$ and $|x-s|=s-x$. The integral equation get the form:

$$\phi(x)+\frac{1}{2}\int_{-1}^{x}(x-s)\,\phi(s)\,\mathrm{d}s+\frac{1}{2}\int_{x}^{1}(s-x)\,\phi(s)\,\mathrm{d}s = x. \tag{14}$$

Taking the derivative of Eq. (14) with respect to x and using the result Eq. (11), we obtain:

$$\phi'(x)+\frac{1}{2}\int_{-1}^{x}\phi(s)\,\mathrm{d}s-\frac{1}{2}\int_{x}^{1}\phi(s)\,\mathrm{d}s=1. \tag{15}$$

Taking the derivative of Eq. (15), we obtain the following ordinary differential equation:

$$\phi''(x)+\phi(x)=0. \tag{16}$$

The general solution of Eq. (16) is as follows:

$$\phi(x)=A\cos(x)+B\sin(x), \tag{17}$$

where A and B are the integration constants. To find them we plug the solution Eq. (17) back into the integral equation and set $x=0$:

$$\phi(0)=A, \tag{18}$$

$$\left(\hat{L}\phi\right)(0)=\frac{1}{2}\int_{-1}^{1}|0-s|\,\phi(s)\,\mathrm{d}s=\frac{A}{2}\int_{-1}^{1}|s|\cos(s)\,\mathrm{d}s+\frac{B}{2}\underbrace{\int_{-1}^{1}|s|\sin(s)\,\mathrm{d}s}_{0}=\alpha A, \tag{19}$$

where

$$\alpha \equiv \int_0^1 s\cos(s)\,\mathrm{d}s. \tag{20}$$

α is positive, since the integrand is positive. (Also, $\alpha = \sin(1) + \cos(1) - 1$ but we do not need the exact value.)

The equation

$$A + \alpha A = 0, \tag{21}$$

where $\alpha > 0$ has the solution

$$A = 0. \tag{22}$$

Thus,

$$\phi(x) = B\sin(x). \tag{23}$$

To determine the constant B we plug the solution Eq. (23) into Eq. (15) and set $x = 1$:

$$\phi'(x) = B\cos(x), \quad \phi'(1) = B\cos(1), \tag{24}$$

$$\frac{1}{2}\int_{-1}^{1}\phi(s)\,\mathrm{d}s = \frac{B}{2}\int_{-1}^{1}\sin(s)\,\mathrm{d}s = 0, \tag{25}$$

$$B\cos(1) = 1, \tag{26}$$

$$B = \frac{1}{\cos(1)}. \tag{27}$$

Therefore, the solution of Eq. (12) is as follows:

$$\phi(x) = \frac{\sin x}{\cos(1)}, \quad -1 \le x \le 1. \tag{28}$$