INDUCED EMF IN A CIRCULAR LOOP

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https://www.phys.uconn.edu/~rozman/Courses/P2400_24S/

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A circular wire loop of radius a has its center at a distance d_0 ($d_0 > a$) from a long straight wire. The wire is in the plane of the loop. (See Fig. 1.) The current in the long wire is changing, $I = I(t)$. What is the induced emf in the loop?

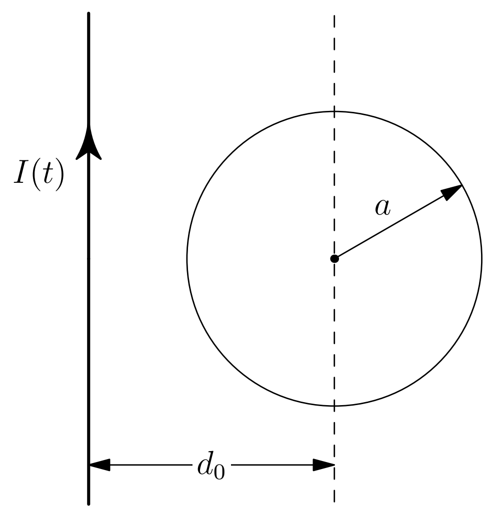


Figure 1: A long straight wire carrying current $I(t)$ and a circular wire loop of radius a with the center distance d_0 from the wire ($d_0 > a$). The wire is in the plane of the loop.

The magnitude of the emf induced in the loop is

$$\varepsilon = \left|\frac{\mathrm{d}\Phi}{\mathrm{d}t}\right|, \tag{1}$$

where Φ is the magnetic flux through the loop,

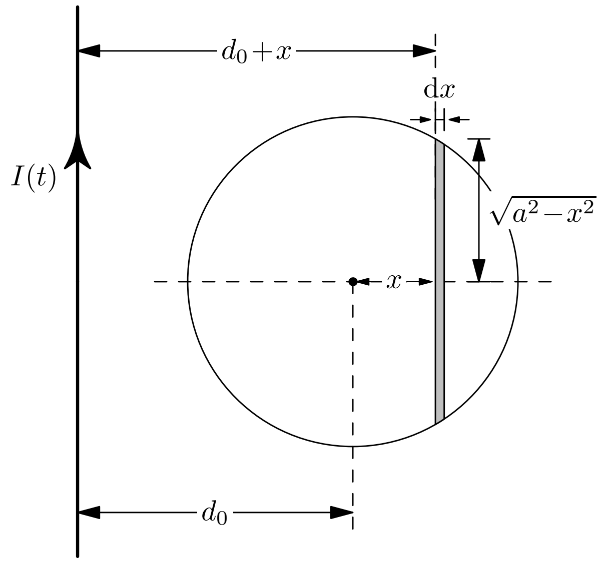
$$\Phi = \int B_n \,\mathrm{d}A, \tag{2}$$

B_n is the component of the magnetic field perpenducular to the plane of the loop; the integration is over the area of the loop. The magnetic field produced by the wire at a distance d from the wire,

$$B_n = \frac{\mu_0 I}{2\pi d}. \tag{3}$$

Calculate emf using cylindrical symmetry of the field

Figure 2: A long straight wire carrying current $I(t)$ and a circular wire loop of radius a with the center distance d_0 from the wire. The wire is in the plane of the loop



Let's use the symmetry of the magnetic field (see Fig. 2) to calculate the magnetic flux Φ, Eq. (2).

The magnitude of the magnetic field depends only on the distance to the wire:

$$B_n(x) = \frac{\mu_0 I}{2\pi(d_0 + x)}. \tag{4}$$

The area element, dA, is as follows:

$$\mathrm{d}A = 2\sqrt{a^2 - x^2}\,\mathrm{d}x. \tag{5}$$

Therefore, the flux through the loop is given by the following expression:

$$\Phi = \int_{-a}^{a} B_n(x)\,\mathrm{d}A = \frac{\mu_0 I}{\pi}\int_{-a}^{a}\frac{\sqrt{a^2 - x^2}}{d_0 + x}\,\mathrm{d}x = \frac{\mu_0 I\, a}{\pi}\int_{-1}^{1}\frac{\sqrt{1 - u^2}}{\beta + u}\,\mathrm{d}u. \tag{6}$$

Here

$$\beta \equiv \frac{d_0}{a}, \quad 1 < \beta < \infty, \tag{7}$$

is the characteristic dimensionless parameter of the problem.

In the last integral we introduced a new integration variable, u:

$$x = a\,u, \quad \mathrm{d}x = a\,\mathrm{d}u, \quad u = \frac{x}{a}, \quad -1 \le u \le 1. \tag{8}$$

Introducing new integration variable, ϕ:

$$u = \sin\phi, \quad \mathrm{d}x = \cos\phi\,\mathrm{d}\phi, \quad -\frac{\pi}{2} \le \phi \le \frac{\pi}{2}, \tag{9}$$

we obtain:

$$\Phi = \frac{\mu_0 I\, a}{\pi} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{\cos^2\phi}{\beta + \sin\phi}\,\mathrm{d}\phi = \frac{\mu_0 I\, a}{2\,\pi} \int_{-\pi}^{\pi} \frac{\cos^2\phi}{\beta + \sin\phi}\,\mathrm{d}\phi. \tag{10}$$

To evaluate the integral in Eq. (10),

$$I(\beta) \equiv \int_{-\pi}^{\pi} \frac{\cos^2\phi}{\beta + \sin\phi}\,\mathrm{d}\phi, \tag{11}$$

let's rewrite it in the form

$$I(\beta) = \frac{1}{2}\,\mathrm{Re}\,\{J(\beta)\}, \tag{12}$$

where

$$J(\beta) \equiv \int_{0}^{2\pi} \frac{e^{2i\phi} + 1}{\beta + \sin\phi}\,\mathrm{d}\phi. \tag{13}$$

Proceeding to the integration in the complex z-plane,

$$z = e^{i\phi}, \quad \mathrm{d}\phi = \frac{\mathrm{d}z}{iz}, \quad \sin\phi = \frac{1}{2i}\left(z - \frac{1}{z}\right), \quad e^{2i\phi} = z^2. \tag{14}$$

The integration contour is the unit circle $|z| = 1$.

$$J(\beta) = \frac{1}{i} \oint_{|z|=1} \frac{z^2 + 1}{\left(\beta + \frac{z}{2i} - \frac{1}{2iz}\right)} \frac{\mathrm{d}z}{z} = 2 \oint_{|z|=1} \frac{z^2 + 1}{z^2 + 2i\beta z - 1}\,\mathrm{d}z \tag{15}$$

The poles of the integrand are given by the roots of the quadratic polynomial in the denominator:

$$z^2 + 2i\beta z - 1 = 0. \tag{16}$$

The roots are:

$$z_{\mathrm{in,out}} = -i\beta \pm i\sqrt{\beta^2 - 1} = i\left(-\beta \pm \sqrt{\beta^2 - 1}\right). \tag{17}$$

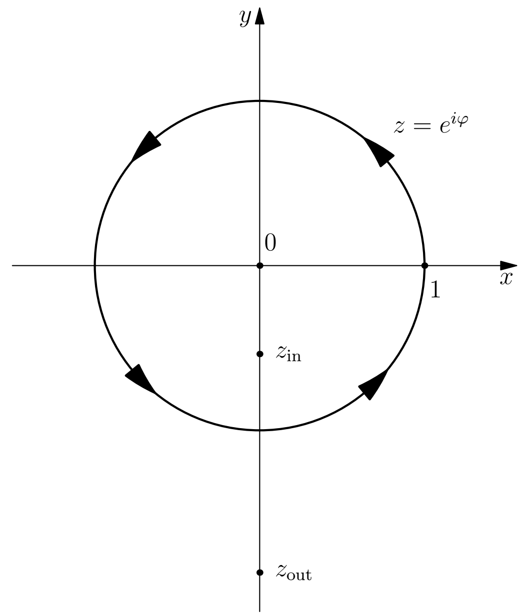


Figure 3: Integration contour for Eq. (15).

z_{in} and z_{out} are imaginary. Since $z_{in}z_{out}$ = -1, only one root, z_{in}, the one with the smaller absolute value, is inside the integration contour:

$$z_{\text{in}} = i\left(\sqrt{\beta^2 - 1} - \beta\right). \tag{18}$$

$$\begin{aligned} J(\beta) &= 4\pi i\,\text{Res}\left(\frac{z^2+1}{(z-z_{\text{in}})(z-z_{\text{out}})}, z = z_{\text{in}}\right) = 4\pi i\frac{z_{\text{in}}^2+1}{z_{\text{in}}-z_{\text{out}}} & (19) \\ &= 2\pi\frac{1-\left(\sqrt{\beta^2-1}-\beta\right)^2}{\sqrt{\beta^2-1}} = 4\pi\left(\beta - \sqrt{\beta^2-1}\right). & (20) \end{aligned}$$

The magnetic flux,

$$\Phi = \mu_0 I\, a\left(\beta - \sqrt{\beta^2-1}\right) = \mu_0 I\, d_0\left(1 - \sqrt{1-\beta^{-2}}\right). \tag{21}$$

Finally, the emf in the loop is

$$\varepsilon = \left|\frac{\mathrm{d}\Phi}{\mathrm{d}t}\right| = \mu_0 d_0\left(1 - \sqrt{1-\beta^{-2}}\right)\left|\frac{\mathrm{d}I}{\mathrm{d}t}\right|. \tag{22}$$

Consistency check

We can verify the result Eq. (21) by analyzing the limit $a \ll d_0$ ($\beta \gg 1$) when the magnetic field through the loop is approximately uniform. We expect the flux to be equal to the product of the area of the loop by the value of the magnetic field in the center of the loop:

$$\Phi \approx \frac{\mu_0 I}{2\pi d_0} \pi a^2 = \frac{\mu_0 I a^2}{2d_0}. \tag{23}$$

Now, if we expand the square root in Eq. (21),

$$\sqrt{1-\beta^{-2}} \approx 1 - \frac{1}{2\beta^2}, \tag{24}$$

we get

$$\Phi = \mu_0 I d_0 \left(1 - \sqrt{1-\beta^{-2}}\right) \approx \mu_0 I d_0 \left[1 - 1 + \frac{1}{2}\frac{a^2}{d_0^2}\right] = \frac{\mu_0 I a^2}{2d_0} \tag{25}$$

as expected.

References

[1] Susan Lea. *Mathematics for Physicists*. Brooks Cole, 2003.